PERGAMON

International Journal of Solids and Structures 38 (2001) 9579–9599

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

# On the modelling of anisotropic elastic and inelastic material behaviour at large deformation

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Received 19 September 1999; in revised form 21 September 2000

**Abstract**

The purpose of this work is the formulation and discussion of an approach to the modelling of anisotropic elastic and inelastic material behaviour at large deformation. This is done in the framework of a thermodynamic, internal-variable-based formulation for such a behaviour. In particular, the formulation pursued here is based on a model for plastic or inelastic deformation as a transformation of local reference configuration for each material element. This represents a slight generalization of its modelling as an elastic material isomorphism pursued in earlier work, allowing one in particular to incorporate the effects of isotropic continuum damage directly into the formulation. As for the remaining deformation- and stress-like internal variables of the formulation, these are modelled in a fashion formally analogous to so-called structure tensors. On this basis, it is shown in particular that, while neither the Mandel nor back stress is generally so, the stress measure thermodynamically conjugate to the plastic "velocity gradient", containing the difference of these two stress measures, *is always symmetric with respect to the Euclidean metric*, i.e., even in the case of classical or induced anisotropic elastic or inelastic material behaviour. Further, in the context of the assumption that the intermediate configuration is materially uniform, it is shown that the stress measure thermodynamically conjugate to the plastic velocity gradient is directly related to the Eshelby stress. Finally, the approach is applied to the formulation of metal plasticity with isotropic kinematic hardening. 

*Keywords:* Thermodynamics; Internal variables; Induced anisotropy; Structure tensors; Large deformation

## 1. Introduction

Since being introduced in various forms by Bilby et al. (1955), Lee and Liu (1967), Lee (1969), and others (e.g., Clifton, 1972), the multiplicative elastoplastic decomposition of the deformation gradient has seen wide application in the formulation of elastoplastic material models for large deformations, in particular those aimed at numerical simulation of such behaviour (e.g., Moran et al., 1990; Eterovic and Bathe, 1990; Weber and Anand, 1990; Simo, 1992; Miehe, 1994). The interpretation of this relation in, and incorporation into, the framework of continuum mechanics, however, has been problematic, if not controversial (see, e.g., Green and Naghdi, 1971; Naghdi, 1990). One purpose of the current work is to show that such a

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0020-7683/01/$ - see front matter 
PII: S0020-7683(01)00140-8

decomposition of the deformation gradient arises naturally in a continuum mechanical setting when the inelastic deformation is modelled as a transformation of local reference placement for each material element. Such a model for the inelastic deformation represents a generalization of earlier work (e.g., Wang and Bloom, 1974; Kratochvíl and Šilhvaý, 1977; Šilhavý and Kratochvíl, 1977; Bertram, 1993; Bertram and Kraska, 1995; Svendsen, 1998) in which the inelastic deformation is modelled as an elastic material isomorphism, i.e., preserves the elastic behaviour of the material. Indeed, modelling the inelastic deformation as a transformation of local reference configuration also includes the case of (isotropic) continuum damage, which results in a degradation of the elastic material properties, and so a change in elastic material behaviour, but (at least in the isotropic case) no change in the form of the elasticity relation.

A second aspect of the current work concerns the modelling of the remaining deformation-like internal variables representing processes such as induced elastic anisotropy or hardening. Following Svendsen (1998) and Dafalias (1998), these are modelled here formally as so-called structure tensors (e.g., Boehler, 1979; Liu, 1982; Zhang and Ryschlewski, 1990; Svendsen, 1994) in the sense that their identification renders the free energy an isotropic function of its arguments. On the other hand, in contrast to classical structure tensors, the deformation-like internal variables evolve, enabling in particular the modelling of induced anisotropic elastic and/or inelastic material behaviour. From this point of view, the deformation-like internal variables characterize the (evolving) material symmetry of the constitutive relations in the formulation. In particular, the classical case of structure tensors representing anisotropic material behaviour is recovered when the internal variables in question are constant.

The paper begins with a summary of the basic constitutive framework utilized in the sequel (Section 2). This consists in particular of a thermodynamic formulation for inelastic material behaviour as based on loading rate and yield functions. In particular, this approach includes standard associated and non-associated thermoelastoplasticity as special cases. Next, the modelling of inelastic deformation as a transformation of local reference configuration is introduced (Section 3). This is followed by a formulation of the evolution relations for this and the other deformation-like internal variables in the context of the dissipation inequality (Section 4). In this context arise as well the stress-like internal variables thermodynamically conjugate to the deformation-like ones, and in particular that associated with the inelastic deformation containing the Mandel and back stresses. Next, the modelling of the internal variables formally as structure tensors is introduced and investigated (Section 5), laying the groundwork for the modelling of elastic and inelastic anisotropic material behaviour. In particular, evolution of the deformation-like internal variables leads in general to induced orthotropic or yet more complex elastic and inelastic anisotropic material behaviour. As shown in previous work (Svendsen, 1998), large-deformation Prager-type (i.e., linear) kinematic hardening models, as well as non-linear models such as those of the Armstrong–Frederick-type, can be formulated within this approach. The next part of the work (Section 6) focuses on the dependence of the material behaviour on the material element in question, i.e., its inhomogeneity, in particular on the special case of materially uniform inhomogeneous behaviour. In the case of crystal plasticity, this assumption is apparently justified; from the general phenomenological point of view, however, it, like material isomorphism, is a true constitutive assumption. In the last part of the work, the approach is applied to the formulation of metal plasticity with isotropic, and Armstrong–Frederick kinematic hardening (Section 7), and the consistent tangent for the numerical implementation of the approach in the finite element method is obtained (Section 8).

Before we begin, a word on notation. If $\mathscr{W}$ and $\mathscr{Z}$ represent two finite-dimensional linear spaces, let $\mathrm{Lin}(\mathscr{W},\mathscr{Z})$ represent the set of all linear mappings from $\mathscr{W}$ to $\mathscr{Z}$. If $\mathscr{W}$ and $\mathscr{Z}$ are inner product spaces, the inner products on $\mathscr{W}$ and $\mathscr{Z}$ induce the transpose $A^{\mathrm{T}} \in \mathrm{Lin}(\mathscr{Z},\mathscr{W})$ of any $A \in \mathrm{Lin}(\mathscr{W},\mathscr{Z})$, as well as the inner product $A \cdot B := \mathrm{tr}_{\mathscr{W}}(A^{\mathrm{T}}B) = \mathrm{tr}_{\mathscr{Z}}(AB^{\mathrm{T}})$ on $\mathrm{Lin}(\mathscr{W},\mathscr{Z})$ for all $A, B \in \mathrm{Lin}(\mathscr{W},\mathscr{Z})$. In particular, let $\mathscr{V}$ represent three-dimensional Euclidean vector space, $\mathrm{Lin} := \mathrm{Lin}(\mathscr{V},\mathscr{V})$ the set of all linear mappings of $\mathscr{V}$ into itself (i.e., second-order Euclidean tensors). Elements of Lin, or time-dependent fields taking values Lin, are represented by bold, upper case italic letters such as $\boldsymbol{A}$, $\boldsymbol{F}$, or $\boldsymbol{T}$, in this work. For any $\boldsymbol{A} \in \mathrm{Lin}$,

let $I_A := \mathrm{tr}(\boldsymbol{A})$ represent its trace, $J_A := \det(\boldsymbol{A})$ its determinant, and $|\boldsymbol{A}| := \sqrt{\boldsymbol{A}\cdot\boldsymbol{A}}$ its magnitude. Further, for any two $\boldsymbol{A}$, $\boldsymbol{B} \in \mathrm{Lin}$, let $[\boldsymbol{A},\boldsymbol{B}] := \boldsymbol{AB} - \boldsymbol{BA}$ represent their Lie bracket, and $\langle \boldsymbol{A},\boldsymbol{B}\rangle := \boldsymbol{AB} + \boldsymbol{BA}$ their Jacobi bracket. Finally, let $\mathrm{sym}(\boldsymbol{A}) := (1/2)(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})$, $\mathrm{skw}(\boldsymbol{A}) := (1/2)(\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}})$, $\mathrm{sph}(\boldsymbol{A}) := (1/3)\boldsymbol{I}_A\boldsymbol{I}$, and $\mathrm{dev}(\boldsymbol{A}) := \boldsymbol{A} - \mathrm{sph}(\boldsymbol{A})$ represent the symmetric, skew-symmetric, spherical, and deviatoric, parts, respectively, of any $\boldsymbol{A} \in \mathrm{Lin}$. For notational simplicity, it proves advantageous to abuse notation in this work and denote mappings and their values by the same symbol. Other notations and mathematical concepts will be introduced as they arise in what follows.

## 2. Basic constitutive framework

As stated in the introduction, the approach pursued here is based on a thermodynamic formulation of inelastic material behaviour as described by yield and loading-rate functions. In particular, the framework to be summarized in what follows extends and generalizes earlier, conceptually similar approaches (e.g., Lubliner, 1973). For simplicity, attention is restricted here to isothermal processes.

Let $B$ represent a material body, $b \in B$ a material element of this body, and $I \subset \mathbb{R}$ a time interval. The "external" mechanical state of any given $b \in B$ is determined in the current constitutive context at any time $t \in I$ by the values $\boldsymbol{F}$ and $\dot{\boldsymbol{F}}$ of the deformation gradient and its rate, respectively, at $(t,b) \in I \times B$. As usual, $\boldsymbol{F}$ and $\dot{\boldsymbol{F}}$ are defined or "measured" with respect to an arbitrary reference configuration of $B$ in $E$. Analogously, the "internal" mechanical state of any $b \in B$ at any $t \in I$ assumed to be determined by the values $\vec{\boldsymbol{\epsilon}} := (\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_n)$ and $\dot{\vec{\boldsymbol{\epsilon}}} := (\dot{\boldsymbol{\epsilon}}_1,\ldots,\dot{\boldsymbol{\epsilon}}_n)$ of $n$ internal variables $\boldsymbol{\epsilon}_i$ and their rates $\dot{\boldsymbol{\epsilon}}_i$ $(i = 1,\ldots,n)$, respectively, again at $(t,b) \in I \times B$. Each of these takes values in some inner product space. The $\vec{\boldsymbol{\epsilon}}$ are modelled as here as *referential* quantities, i.e., quantities defined with respect to the above-mentioned arbitrary reference configuration of $B$.

Turning next to the constitutive relations, we have in particular the basic material frame-indifferent forms

$$\begin{aligned} \psi &= \psi(\boldsymbol{C},\vec{\boldsymbol{\epsilon}},\dot{\boldsymbol{C}},b),\\ \boldsymbol{S} &= \boldsymbol{S}(\boldsymbol{C},\vec{\boldsymbol{\epsilon}},\dot{\boldsymbol{C}},b), \end{aligned} \tag{2.1}$$

of the referential free energy density $\psi$ and second Piola–Kirchhoff stress $\boldsymbol{S}$ for the material element $b \in B$. These are in general functions of the right Cauchy–Green deformation $\boldsymbol{C} := \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$ and its rate $\dot{\boldsymbol{C}}$, as well as $\vec{\boldsymbol{\epsilon}}$. The basic constitutive relations are completed by the form

$$\dot{\boldsymbol{\epsilon}}_i = \dot{\boldsymbol{\epsilon}}_i(\boldsymbol{C},\vec{\boldsymbol{\epsilon}},\dot{\boldsymbol{C}},b) \tag{2.2}$$

for the evolution of the internal variables. In the current formulation, the form of this last relation is determined in particular by a yield function $\phi = \phi(\boldsymbol{C},\vec{\boldsymbol{\epsilon}},b)$ and a so-called loading rate function $\ell = \ell(\boldsymbol{C},\vec{\boldsymbol{\epsilon}},\dot{\boldsymbol{C}},b)$ (e.g., Lubliner, 1973). In particular, the values $\ell < 0$, $\ell = 0$ and $\ell > 0$, respectively, of $\ell$ determine unloading, neutral, and loading, conditions, respectively.

In context of rate-independent material behaviour, Eq. (2.1) are as usual positive degree zero, while Eq. (2.2) and $\ell(\boldsymbol{C},\vec{\boldsymbol{\epsilon}},\dot{\boldsymbol{C}},b)$ are positive degree one, homogeneous functions of $\dot{\boldsymbol{C}}$. Assuming further that $\ell(\boldsymbol{C},\vec{\boldsymbol{\epsilon}},\cdot,b)$ is continuously differentiable (at least near 0), it reduces to

$$\ell(\boldsymbol{C},\vec{\boldsymbol{\epsilon}},\dot{\boldsymbol{C}},b) = \boldsymbol{N}(\boldsymbol{C},\vec{\boldsymbol{\epsilon}},b)\cdot\dot{\boldsymbol{C}}. \tag{2.3}$$

Likewise, assuming [1] that Eq. (2.1) are continuous in $\dot{\boldsymbol{C}}$, they simplify to

[1] In the more recent literature, Antman (1995) [Chapter XV, Sections 2 and 3] utilized similar technical assumptions on the stress, free energy and internal variable evolution constitutive relations to formulate thermoelastoplasticity based on a yield function, representing a special case of the current formulation as discussed in the text.

$$\psi = \psi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, b), \tag{2.4a}$$

$$\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, b), \tag{2.4b}$$

again in the rate-independent case. Lastly, the fact that the material behaviour, and in particular the evolution of the $\vec{\boldsymbol{\epsilon}}$, is different at loading and unloading conditions, is accounted for here by modelling $\dot{\boldsymbol{\epsilon}}_i(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, \cdot, b)$ as piecewise continuously differentiable on the domain of $\ell(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, \cdot, b)$. When we combine this with the usual assumption that the $\vec{\boldsymbol{\epsilon}}$ evolve only for $\phi = 0$ and $\ell > 0$ (i.e., under loading conditions), Eq. (2.2) reduces to

$$\dot{\boldsymbol{\epsilon}}_i(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, \dot{\boldsymbol{C}}, b) = h(\phi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, b))\langle \ell(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, \dot{\boldsymbol{C}}, b)\rangle \boldsymbol{\varpi}_{\epsilon_i}(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, b), \tag{2.5}$$

again in the rate-independent contex. Here,

$$\langle x\rangle := \tfrac{1}{2}(x + |x|) = h(x)x \tag{2.6}$$

represents the MacCauley bracket, and $h(x)$ the modified step function (i.e., $h(x < 0) = 0$, $h(x \geqslant 0) = 1$). Note that the formulation to follow can be used to model rate-dependent (i.e., viscoplastic) material behaviour as well when we *formally* replace $h(\phi)\langle \ell\rangle$ appearing in Eq. (2.5) by, e.g., a function of $\langle \phi\rangle$, and extend the states of the system to $\phi > 0$. Further, note that Eqs. (2.1)–(2.5), as well as all other constitutive relations to follow in this section, are expressed with respect to the given but arbitrary reference configuration of the material (body) in question. Since we do not deal with the dependence of the constitutive relations on $b$ until Section 6, we dispense with it in the notation for simplicity until then.

Turning next to thermodynamic considerations, these are based as usual on the referential form

$$\delta := \tfrac{1}{2}\boldsymbol{S} \cdot \dot{\boldsymbol{C}} - \dot{\psi} \tag{2.7}$$

of the dissipation rate density $\delta$ in the current isothermal context, representing the difference between work performed on, and the work stored in, the material per unit volume and time. Substituting now Eqs. (2.4a) and (2.5) into Eq. (2.7), it reduces to

$$\delta = \left[\tfrac{1}{2}\boldsymbol{S} - \psi_{,\boldsymbol{C}} - h(\phi)h(\boldsymbol{N} \cdot \dot{\boldsymbol{C}}) \sum_{i=1}^{n} (\psi_{,\epsilon_i} \cdot \boldsymbol{\varpi}_{\epsilon_i})\boldsymbol{N}\right] \cdot \dot{\boldsymbol{C}} \tag{2.8}$$

via Eqs. (2.3) and (2.6). For the current constitutive class, then, $\delta(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, \dot{\boldsymbol{C}})$ is positive homogeneous of degree one in $\dot{\boldsymbol{C}}$; in fact, it is piecewise linear in $\dot{\boldsymbol{C}}$. Consequently, the usual hyperelastic form

$$\boldsymbol{S} = 2\psi_{,\boldsymbol{C}} \tag{2.9}$$

for $\boldsymbol{S}$ is not in general necessary to satisfy the Coleman–Noll dissipation principle (see, e.g., Coleman and Gurtin, 1967, for this principle) in the more general elastoplastic case. On the other hand, Eq. (2.9) is *necessary* to insure that elastic processes (i.e., those for which $\phi < 0$) result in no internal dissipation, i.e., to insure that

$$\delta|_{\phi<0} = \left[\tfrac{1}{2}\boldsymbol{S} - \psi_{,\boldsymbol{C}}\right] \cdot \dot{\boldsymbol{C}} = 0 \tag{2.10}$$

from Eq. (2.8) *holds identically*. To reiterate, the hyperelastic form Eq. (2.9) results from the constitutive assumption Eq. (2.10) reflecting the physical nature of elastic processes, and not as any kind of restriction

from the dissipation principle and Clausius–Duhem inequality $\delta \geqslant 0$, in the more general rate-independent inelastic context. In the context of Eq. (2.10), Eq. (2.8) simplifies yet further to

$$\delta = h(\phi)\langle \ell \rangle \sum_{i=1}^{n} \boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i} \cdot \boldsymbol{\sigma}_i \tag{2.11}$$

for $\delta = \delta(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, \dot{\boldsymbol{C}})$ via Eq. (2.5), with

$$\boldsymbol{\sigma}_i := -\psi_{,\boldsymbol{\epsilon}_i} \tag{2.12}$$

the quantity thermodynamically conjugate to $\boldsymbol{\epsilon}_i$.

Assuming forms for $\psi$ and $\phi$ have been established, the above formulation is complete once we have formulated those for the constitutive quantity $\boldsymbol{N}$ determining $\ell$ via Eq. (2.3), and for that $\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i}$ determining together with $\ell$ the form of the evolution relation for $\boldsymbol{\epsilon}_i$. In the context of a rate-based formulation, for example, one can determine the form of $\boldsymbol{N}$ with the help of the so-called consistency condition $\dot{\phi} = 0$ at states where $\phi = 0$. Indeed, via Eqs. (2.5) and (2.14), this condition is satisfied identically at such states when

$$\boldsymbol{N} = \frac{-\phi_{,\boldsymbol{C}}}{\sum_{i=1}^{n} \phi_{,\boldsymbol{\epsilon}_i} \cdot \boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i}} \tag{2.13}$$

holds. Alternatively, one could work with the constitutive assumption

$$\boldsymbol{N} = \phi_{,\boldsymbol{C}} \tag{2.14}$$

for $\boldsymbol{N}$. This assumption reduces the current approach to one compatible with that of Antman (1995, Chapter XV, Section 2), who worked with the form Eq. (2.14) for $\boldsymbol{N}$ from the start. Note that this last form for $\boldsymbol{N}$ is consistent with $\dot{\phi} = 0$ at $\phi = 0$ when the constraint

$$\sum_{i=1}^{n} \phi_{,\boldsymbol{\epsilon}_i} \cdot \boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i} = -1 \tag{2.15}$$

on the forms of $\phi$ and $\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_1}, \ldots, \boldsymbol{\varpi}_{\boldsymbol{\epsilon}_n}$ holds. In either case, $\boldsymbol{N}$, and so $\ell$ via Eq. (2.3), is determined by the form of $\phi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}})$ as well as the evolution of the $\vec{\boldsymbol{\epsilon}}$. Lastly, in a non-rate-based formulation, rather than determining $\boldsymbol{N}$ explicitly, note that one simply identifies $h(\phi)\langle \ell \rangle$ with the standard plastic multiplier $\lambda$ whose value is determined via enforcement of the yield condition $\phi = 0$ during inelastic processes.

Up to this point, we have been dealing tacitly with the so-called "strain-space" or "deformation-space" formulation of elastoplasticity or viscoplasticity. To obtain the corresponding "stress-space" formulation in the current framework, one assumes that the yield function $\phi = \phi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}})$ depends on $\boldsymbol{C}$ only through the dependence of $\vec{\boldsymbol{\sigma}}$ on $\boldsymbol{C}$ via Eqs. (2.4a) and (2.12). In other words, there exists a constitutive function $\phi_{\mathrm{S}} = \phi_{\mathrm{S}}(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\epsilon}})$ such that

$$\phi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}) = \phi_{\mathrm{S}}(\vec{\boldsymbol{\sigma}}(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}), \vec{\boldsymbol{\epsilon}}) \tag{2.16}$$

holds. In addition, $\Phi_{\mathrm{S}}(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\epsilon}})$ is assumed continuously differentiable in $\vec{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n)$. This completes our brief summary of the basic constitutive and thermodynamic framework required for the sequel. Now, we turn to the modeling of inelastic deformation.

## 3. Plastic transformation as a change of local reference configuration

A basic aspect of the continuum modelling of material behaviour is the dependence of the constitutive relations on a given but otherwise arbitrary reference configuration of the material body in question. In particular, all constitutive relations considered up to this point are based on such a given but otherwise arbitrary reference configuration of $B$ in $E$ and the corresponding *placement* (i.e., mapping) of $B$ into this configuration. In the context of simple material behaviour, the dependence of the constitutive relations on reference configuration becomes one on the corresponding *local* reference placement or configuration [2] at each material element (e.g., Truesdell and Noll, 1965 [Section 28]; Noll, 1967, 1972). For comparison, note that $\boldsymbol{F}$ deforms the corresponding local reference configuration of $b$ into its local current configuration, i.e., that in the current configuration of $B$.

Consider now a second local reference configuration of $b$, and let $\boldsymbol{H}$ represent the transformation of the original or first local reference configuration into the second. Since the material behaviour of any $b \in B$ does not depend on the choice of local reference configuration used to represent it, such a transformation induces in particular those

$$\psi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}) = J_{\boldsymbol{H}} \psi_{\bigstar}(\boldsymbol{H}_* \boldsymbol{C}, \boldsymbol{H}_* \vec{\boldsymbol{\epsilon}}) \tag{3.1a}$$

$$\phi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}) = \phi_{\bigstar}(\boldsymbol{H}_* \boldsymbol{C}, \boldsymbol{H}_* \vec{\boldsymbol{\epsilon}}) \tag{3.1b}$$

$$\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i}(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}) = \boldsymbol{H}^* [\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i \bigstar}(\boldsymbol{H}_* \boldsymbol{C}, \boldsymbol{H}_* \vec{\boldsymbol{\epsilon}})] \tag{3.1c}$$

of the referential free energy density $\psi$, as well as the inelastic constitutive quantities $\phi$ and $\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i}$ determining the evolution of $\boldsymbol{\epsilon}_i$ with respect to the first local reference configuration into those $\psi_{\bigstar}$, $\phi_{\bigstar}$ and $\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i \bigstar}$, respectively, relative to the second. Here, $\boldsymbol{H}_*$ signifies the "push-forward" operation induced on (referential) Euclidean tensors by $\boldsymbol{H}$; in particular, we have $\boldsymbol{H}_* \boldsymbol{C} = \boldsymbol{H}^{-\mathrm{T}} \boldsymbol{C} \boldsymbol{H}^{-1}$. Note that Eq. (3.1a) implies that $\psi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}})$ and $\psi_{\bigstar}(\boldsymbol{H}_* \boldsymbol{C}, \boldsymbol{H}_* \vec{\boldsymbol{\epsilon}})$ have the *same form* with respect to *different arguments*; the same applies for the other constitutive quantities as well. Consequently, Eqs. (3.1a)–(3.1c) represents in a sense a generalization of Noll's concept of material isomorphism (Noll, 1958, Section 14; see also Truesdell and Noll, 1965, Section 27, or Noll, 1967, 1972) with respect to change of local reference configuration. Indeed, phrasing this notion with respect to two distinct local reference configuration of the same material element, rather than with respect to two distinct material elements, Eqs. (3.1a)–(3.1c) implies that any unimodular transformation $\boldsymbol{H}$ of local reference configuration is such an isomorphism (i.e., in this alternative sense). For general transformations $\boldsymbol{H}$, this is, strictly speaking, no longer the case. As such, the modelling of inelastic deformation formally as a transformation of local reference configuration to follow represents a slight generalization of its modelling as an (elastic) material isomorphism. In particular, this allows the inclusion of (isotropic) continuum damage in the formulation, as discussed below.

From a purely phenomenological point of view, having no idea about any internal "structure" the material might possess, our knowledge of inelastic processes is in effect limited to the observation that these lead to persistent [3] change of shape of the material, and possibly result in a change in material behaviour

[2] A local placement of $b$, and so the corresponding configuration, is induced by an equivalence class of global placements of $B$ into $E$ at $b \in B$. As recognized by Noll (1958), two global placements of $B$ are equivalent at $b \in B$ when their relative gradient at $b \in B$ is equal to the identity tensor. Note that local placements of $b$ map an infinitesimal neighborhood of $b$ in $B$ (i.e., the tangent space to $B$ at $b$) into the corresponding local configuration of $b$. By definition, this local configuration of $b$ is induced by any global placement of $B$ belonging to the equivalence class generating the local placement in question.

[3] In the current isothermal context, one could perhaps use "permanent" instead of "persistent" here; in the more general thermodynamic context, however, there always exist processes (e.g., melting) which can revert the shape of the material to its original form. From this point of view, "persistent" would seem a more appropriate designation in general.

(e.g., induced elastic anisotropy, hardening, and so on). On this basis, the set $\vec{\boldsymbol{\epsilon}}$ of deformation-like internal variables should include elements modelling the corresponding transformation of (local) reference configuration as well as the effect of any further inelastic processes on the overall material behaviour. Here we are assuming in particular that a persistent change of shape can be associated with a transformation of reference configuration, or, in the context of simple material behaviour, a transformation of local reference configuration. [4] To this end, let the last element $\boldsymbol{\epsilon}_n$ of the set $\vec{\boldsymbol{\epsilon}}$ of internal variables represent a time-dependent invertible second-order tensor $\boldsymbol{P}$ with positive determinant, representing a transformation of the *local* reference configuration of $b \in B$ induced by inelastic processes. In the context of phenomenological elastoplasticity or viscoplasticity with isotropic damage, for example, one often associates the volumetric part $J_{\boldsymbol{P}}^{1/3}\boldsymbol{I}$ of $\boldsymbol{P}$ with persistent change of shape due to damage, and the unimodular part $\boldsymbol{P}/J_{\boldsymbol{P}}^{1/3}$ with that due to dislocation/defect motion, e.g., in glide planes and/or on grain boundaries. The corresponding damage parameter can be defined as

$$d := 1 - J_{\boldsymbol{P}}^{-1}. \tag{3.2}$$

So, on this basis, $\vec{\boldsymbol{\epsilon}}$ is split into the pair $(\vec{\boldsymbol{\alpha}}, \boldsymbol{P})$, the remaining internal quantities $\vec{\boldsymbol{\alpha}} := (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n-1}) \mathrel{\widehat{=}} (\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_{n-1})$ account for the effect of inelastic processes such as hardening or induced elastic anisotropy on the material behaviour of $b$ via the dependence of $\psi$ on these.

By formal analogy with Eqs. (3.1a)–(3.1c), then, the modelling of $\boldsymbol{P}$ as a transformation of the *local* reference configuration of $b \in B$ induced by inelastic processes leads to the reduced forms

$$\psi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}) = J_{\boldsymbol{P}}\psi_{\bigstar}(\boldsymbol{P}_*\boldsymbol{C}, \boldsymbol{P}_*\vec{\boldsymbol{\alpha}}, \boldsymbol{P}_*\boldsymbol{P}) = J_{\boldsymbol{P}}\psi_{\bigstar}(\boldsymbol{P}_*\boldsymbol{C}, \boldsymbol{P}_*\vec{\boldsymbol{\alpha}}, \boldsymbol{I}) \tag{3.3a}$$

$$\phi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}) = \phi_{\bigstar}(\boldsymbol{P}_*\boldsymbol{C}, \boldsymbol{P}_*\vec{\boldsymbol{\alpha}}, \boldsymbol{P}_*\boldsymbol{P}) = \phi_{\bigstar}(\boldsymbol{P}_*\boldsymbol{C}, \boldsymbol{P}_*\vec{\boldsymbol{\alpha}}, \boldsymbol{I}) \tag{3.3b}$$

$$\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i}(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}) = \boldsymbol{P}^*[\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i\bigstar}(\boldsymbol{P}_*\boldsymbol{C}, \boldsymbol{P}_*\vec{\boldsymbol{\alpha}}, \boldsymbol{P}_*\boldsymbol{P})] = \boldsymbol{P}^*[\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i\bigstar}(\boldsymbol{P}_*\boldsymbol{C}, \boldsymbol{P}_*\vec{\boldsymbol{\alpha}}, \boldsymbol{I})] \tag{3.3c}$$

for the basic depend constitutive quantities via $\boldsymbol{P}_*\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}^{-1} = \boldsymbol{I}$, where $\boldsymbol{P}_*$ represents the "push-forward" operator induced by $\boldsymbol{P}$ on Euclidean tensors. In fact, it proves useful to work with the slight generalizations

$$\psi(\boldsymbol{C}, \vec{\boldsymbol{\alpha}}, \boldsymbol{P}) = \varphi(\boldsymbol{P}_*\boldsymbol{C}, \boldsymbol{P}_*\vec{\boldsymbol{\alpha}}, J_{\boldsymbol{P}}), \tag{3.4a}$$

$$\phi(\boldsymbol{C}, \vec{\boldsymbol{\alpha}}, \boldsymbol{P}) = \chi(\boldsymbol{P}_*\boldsymbol{C}, \boldsymbol{P}_*\vec{\boldsymbol{\alpha}}, J_{\boldsymbol{P}}), \tag{3.4b}$$

$$\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i}(\boldsymbol{C}, \vec{\boldsymbol{\alpha}}, \boldsymbol{P}) = \boldsymbol{P}^*[\boldsymbol{\pi}_{\boldsymbol{P}_*\boldsymbol{\epsilon}_i}(\boldsymbol{P}_*\boldsymbol{C}, \boldsymbol{P}_*\vec{\boldsymbol{\alpha}}, J_{\boldsymbol{P}})], \tag{3.4c}$$

of Eqs. (3.3a)–(3.3c) in what follows. As evident, Eqs. (3.4a)–(3.4c) reduces to Eqs. (3.3a)–(3.3c) when $\varphi$ is given by $J_{\boldsymbol{P}}\psi_{\bigstar}$, and when $\chi$ and $\boldsymbol{\pi}_{\boldsymbol{P}_*\boldsymbol{\epsilon}_i}$ are independent of $J_{\boldsymbol{P}}$. In any case, the modelling of $\boldsymbol{P}$ as such a transformation of local reference configuration leads to a form for the referential free energy density $\psi$ which depends *explicitly* on $\boldsymbol{P}$ only through $J_{\boldsymbol{P}}$.

An immediate consequence of Eq. (3.4a) is the form

$$\boldsymbol{S} = 2\psi_{,\boldsymbol{C}} = 2\boldsymbol{P}^{-1}\varphi_{,\boldsymbol{P}_*\boldsymbol{C}}\boldsymbol{P}^{-\mathrm{T}} \tag{3.5}$$

for the second Piola–Kirchhoff stress with respect to the arbitrary local reference configuration via Eq. (2.9). Likewise, the form

[4] In general, two or more local configurations are required when the internal structure involved evolves with respect to the deformation of the entire body. Examples of this include the "substructure" of Dafalias (1985); more recently, Wineman and Rajagopal (1990), and Rajagopal and Wineman (1992) have investigated models of inelastic behaviour involving two or more reference configurations, e.g., in the case of twinning, one for each twin. As such, three or more such configurations are involved in the formulation of the model.

$$J_{\boldsymbol{F}}^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{\mathrm{T}} = 2(1-d)J_{\boldsymbol{F}\boldsymbol{P}^{-1}}^{-1}(\boldsymbol{F}\boldsymbol{P}^{-1})\varphi_{,\boldsymbol{P}_*\boldsymbol{C}}(\boldsymbol{F}\boldsymbol{P}^{-1})^{\mathrm{T}} \tag{3.6}$$

then holds for the Cauchy stress via Eq. (3.2). Since $\boldsymbol{P}_*\boldsymbol{C} = \boldsymbol{P}^{-\mathrm{T}}\boldsymbol{C}\boldsymbol{P}^{-1} = (\boldsymbol{F}\boldsymbol{P}^{-1})^{\mathrm{T}}(\boldsymbol{F}\boldsymbol{P}^{-1})$, one sees that the modelling $\boldsymbol{P}$ as a transformation of local reference configuration, i.e., Eqs. (3.4a)–(3.4c) leads to a dependence of the effective Cauchy stress Eq. (3.6) in particular on the combination $\boldsymbol{F}\boldsymbol{P}^{-1}$ of $\boldsymbol{F}$ and $\boldsymbol{P}$, relative to the transformed local reference configuration. From the point of view that deformation resulting in a change of the effective stress is "elastic " in nature, then, the combination [5] $\boldsymbol{F}\boldsymbol{P}^{-1}$ represents such an "elastic deformation," or the "elastic part" of $\boldsymbol{F}$, and

$$\boldsymbol{C}_{\mathrm{E}} = \boldsymbol{P}^{-\mathrm{T}}\boldsymbol{C}\boldsymbol{P}^{-1} = \boldsymbol{P}_*\boldsymbol{C} \tag{3.7}$$

the corresponding right Cauchy–Green deformation measure. On this basis, $\boldsymbol{F}\boldsymbol{P}^{-1}$ can be identified with the standard elastic "deformation gradient" $\boldsymbol{F}_{\mathrm{E}}$, and $\boldsymbol{P}$ with its plastic counterpart $\boldsymbol{F}_{\mathrm{P}}$, from the standard multiplicative decomposition $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{E}}\boldsymbol{F}_{\mathrm{P}}$ (e.g., Bilby et al., 1955; Lee and Liu, 1967; Lee, 1969) of $\boldsymbol{F}$ in the context of crystal plasticity, and its phenomenological generalization (e.g., Sidoroff, 1973; Haupt, 1985; Dashner, 1993; Agah-Tehrani et al., 1987; Lubarda, 1991). Consequently, the modelling of $\boldsymbol{P}$ as a transformation of local reference configuration due to inelastic processes in the material, i.e., Eqs. (3.4a)–(3.4c), induces, or is at least consistent with, the multiplicative elastoplastic decomposition of $\boldsymbol{F}$.

## 4. Formulation relative to the transformed local reference configuration

The modelling of $\boldsymbol{P}$ via Eqs. (3.4a)–(3.4c) has a number of consequences for the thermodynamic formulation in the framework of the referential dissipation rate density $\delta$ as given in Eq. (2.11). To investigate these, consider the form

$$\delta = \sum_{i=1}^{n} \boldsymbol{\sigma}_i\dot{\boldsymbol{\epsilon}}_i = \sum_{i=1}^{n-1} \boldsymbol{\kappa}_i\dot{\boldsymbol{\alpha}}_i + \boldsymbol{\Sigma}\cdot\dot{\boldsymbol{P}} \tag{4.1}$$

of Eq. (2.11) for $\delta$ obtained from Eqs. (2.5) and (2.12), $\vec{\boldsymbol{\epsilon}} = (\vec{\boldsymbol{\alpha}}, \boldsymbol{P})$ and $\vec{\boldsymbol{\sigma}} = (\vec{\boldsymbol{\kappa}}, \boldsymbol{\Sigma})$, with

$$\boldsymbol{\kappa}_i := -\psi_{,\boldsymbol{\alpha}_i} \tag{4.2}$$

the stress-like internal variable thermodynamically conjugate to $\boldsymbol{\alpha}_i$, and

$$\boldsymbol{\Sigma} := -\psi_{,\boldsymbol{P}} \tag{4.3}$$

that conjugate to $\boldsymbol{\epsilon}_n = \boldsymbol{P}$ $(\boldsymbol{\sigma}_n = \boldsymbol{\Sigma})$ from Eq. (2.12). Relative to the transformed local reference configuration, these take the forms

$$\boldsymbol{\kappa}_{\mathrm{I}i} := -\varphi_{,\boldsymbol{\alpha}_{\mathrm{I}i}} = \boldsymbol{P}_*^{-\mathrm{T}}\boldsymbol{\kappa}_i, \tag{4.4}$$

with

$$\boldsymbol{\alpha}_{\mathrm{I}i} := \boldsymbol{P}_*\boldsymbol{\alpha}_i, \tag{4.5}$$

and

$$\boldsymbol{\Sigma}_{\mathrm{I}} := -\psi_{,\boldsymbol{P}}\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{P}_*^{-\mathrm{T}}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{M} - \boldsymbol{X} \tag{4.6}$$

via Eq. (3.4a). Here,

[5] On the basis of the modelling of $\boldsymbol{P}$ as a transformation of local configuration, note that $\boldsymbol{F}\boldsymbol{P}^{-1}$ represents in effect the deformation gradient with respect to the transformed reference local configuration.

$$\boldsymbol{M} := 2\boldsymbol{C}_{\mathrm{E}}\varphi_{,\boldsymbol{C}_{\mathrm{E}}} \tag{4.7}$$

represents Mandel's (1971, 1974) stress tensor (see, e.g., Lubliner, 1986) via Eq. (3.7),

$$\boldsymbol{X} := \sum_{i=1}^{n-1}(\boldsymbol{P}_{*}\boldsymbol{\alpha}_i)^{\mathrm{T}}_{,\boldsymbol{P}}[\varphi_{,\boldsymbol{P}_{*}\boldsymbol{\alpha}_i}]\boldsymbol{P}^{\mathrm{T}} + J_{\boldsymbol{P}}\varphi_{,J_{\boldsymbol{P}}}\boldsymbol{I} \tag{4.8}$$

the contribution to $\boldsymbol{\Sigma}_{\mathrm{I}}$ from inelastic processes. As will become evident later, $\boldsymbol{X}$ appearing in Eq. (4.6) can be identified with the center of the elastic range of the material, i.e., with the back stress. And as discussed in Section 6, $\boldsymbol{\Sigma}_{\mathrm{I}}$ itself is related to the so-called Eshelby stress in the material with respect to the transformed local reference configuration.

In terms of the quantities $\vec{\boldsymbol{\alpha}}_{\mathrm{I}}$, $\vec{\boldsymbol{\kappa}}_{\mathrm{I}}$ and $\boldsymbol{\Sigma}_{\mathrm{I}}$, the dissipation rate density takes the form

$$\delta = \sum_{i=1}^{n-1}\boldsymbol{\kappa}_{\mathrm{I}i}\overset{\Delta}{\boldsymbol{\alpha}}_{\mathrm{I}i} + \boldsymbol{\Sigma}_{\mathrm{I}}\cdot\boldsymbol{L}_{\mathrm{P}} \tag{4.9}$$

relative to the transformed local reference configuration via Eqs. (4.1), (4.2) and (4.5). Here,

$$\boldsymbol{L}_{\mathrm{P}} := \boldsymbol{P}_{*}\dot{\boldsymbol{P}} = \dot{\boldsymbol{P}}\boldsymbol{P}^{-1} \tag{4.10}$$

represents the plastic "velocity gradient," and

$$\overset{\Delta}{\boldsymbol{\alpha}}_{\mathrm{I}i} := \boldsymbol{P}_{*}\dot{\boldsymbol{\alpha}}_i = \boldsymbol{P}_{*}(\boldsymbol{P}^{*}\boldsymbol{\alpha}_{\mathrm{I}i}) \tag{4.11}$$

the evolution of $\boldsymbol{\alpha}_{\mathrm{I}i}$ with respect to the "flow" of $\boldsymbol{P}$. In addition, the evolution relation

$$\dot{\boldsymbol{\alpha}}_i = h(\phi)\langle\ell\rangle\boldsymbol{\varpi}_{\boldsymbol{\alpha}_i} \tag{4.12}$$

for each $\boldsymbol{\alpha}_i (i = 1,\ldots,n-1)$ becomes

$$\overset{\Delta}{\boldsymbol{\alpha}}_{\mathrm{I}i} = h(\chi)\langle m\rangle\boldsymbol{\pi}_{\boldsymbol{\alpha}_{\mathrm{I}i}} \tag{4.13}$$

relative to the transformed local reference configuration via Eqs. (3.4b) and (3.4c). Here, $m$ is reduced form of $\ell$ relative to this configuration. Likewise, the evolution relation

$$\dot{\boldsymbol{P}} = h(\phi)\langle\ell\rangle\boldsymbol{\varpi}_{\boldsymbol{P}} \tag{4.14}$$

for $\boldsymbol{P}$ from Eq. (2.5) takes the form

$$\boldsymbol{L}_{\mathrm{P}} = h(\chi)\langle m\rangle\boldsymbol{\pi}_{\boldsymbol{I}}, \tag{4.15}$$

relative to the transformed local reference configuration, again via Eqs. (3.4a) and (3.4b).

## 5. Internal variables and anisotropy

With the model for $\boldsymbol{P}$ as a transformation of local reference configuration in hand, we now turn to that for the deformation-like internal variables $\vec{\boldsymbol{\alpha}}$. By definition, these represent the effects of inelastic processes such as hardening on the material behaviour. In the context of Eqs. (3.4a)–(3.4c), these and other such possibilities can be accounted for in the simplest fashion when the $\vec{\boldsymbol{\alpha}}_{\mathrm{I}}$ are modelled formally as so-called structure tensors for $\varphi$ (e.g., Boehler, 1979; Liu, 1982; Zhang and Ryschlewski, 1990; Svendsen, 1994). By definition, these represent such tensors for $\varphi$ when

$$\varphi(\boldsymbol{C}_{\mathrm{E}}, \vec{\boldsymbol{\alpha}}_{\mathrm{I}}, J_{\boldsymbol{P}}) = \varphi(\boldsymbol{Q}_{*}\boldsymbol{C}_{\mathrm{E}}, \boldsymbol{Q}_{*}\vec{\boldsymbol{\alpha}}_{\mathrm{I}}, J_{\boldsymbol{QP}}) \tag{5.1}$$

holds for all rotations $\boldsymbol{Q}$, i.e., when $\varphi(\boldsymbol{C}_{\mathrm{E}}, \vec{\boldsymbol{\alpha}}_{\mathrm{I}}, J_{\boldsymbol{P}})$ is an isotropic function of its arguments. Now, since Eq. (5.1) holds for all rotations $\boldsymbol{Q}$, it holds in particular for any curve $\boldsymbol{Q}(s) = \exp(-s\boldsymbol{\Omega})$ generated by some spin

$\boldsymbol{\Omega}$. Substituting this choice for $\boldsymbol{Q}$ into Eq. (5.1), taking the derivative the resulting function of $s$, and evaluating the result at $s = 0$, yields the constraint [6]

$$\boldsymbol{\Sigma}_{\mathrm{I}} \cdot \boldsymbol{\Omega} = 0 \tag{5.2}$$

on the form of $\varphi$ in the guise of $\boldsymbol{\Sigma}_{\mathrm{I}}$ via Eqs. (3.7) and (4.5), the chain rule, and Eqs. (4.6)–(4.8). Since $\boldsymbol{\Omega}$ is arbitrary, Eq. (5.2) can be satisfied only if $\boldsymbol{\Sigma}_{\mathrm{I}}$ is *symmetric* (i.e., with respect to the Euclidean metric). On the basis of Eq. (4.6), this implies that, although both the Mandel stress $\boldsymbol{M}$ and back stress $\boldsymbol{X}$ are in general not symmetric, their difference is $\boldsymbol{M} - \boldsymbol{X}$. We come back to this below.

A particular consequence of the symmetry of $\boldsymbol{\Sigma}_{\mathrm{I}}$ implied by Eq. (5.2) is the reduction of the dissipation rate density $\delta$ as given by Eq. (4.9) to the form

$$\delta = \sum_{i=1}^{n-1} \boldsymbol{\kappa}_{\mathrm{I}i} \overset{\Delta}{\vec{\boldsymbol{\alpha}}}_{\mathrm{I}i} + \boldsymbol{\Sigma}_{\mathrm{I}} \cdot \boldsymbol{D}_{\mathrm{P}}, \tag{5.3}$$

with

$$\boldsymbol{D}_{\mathrm{P}} := \mathrm{sym}(\boldsymbol{L}_{\mathrm{P}}) = h(\chi)\langle m \rangle \mathrm{sym}(\boldsymbol{\pi}_{\boldsymbol{I}}) \tag{5.4}$$

the continuum or material plastic deformation rate from Eq. (4.15). A term of the form $\boldsymbol{\Sigma}_{\mathrm{I}} \cdot \boldsymbol{D}_{\mathrm{P}}$ in $\delta$ is well known in the case of isotropic material behaviour; here, we have shown that it arises in the anisotropic case when the $\vec{\boldsymbol{\alpha}}$ are modelled as evolving structure tensors. As such, even in the more general anisotropic case, the constitutive form

$$\boldsymbol{W}_{\mathrm{P}} := \mathrm{skw}(\boldsymbol{L}_{\mathrm{P}}) = h(\chi)\langle m \rangle \mathrm{skw}(\boldsymbol{\pi}_{\boldsymbol{I}}) \tag{5.5}$$

for the continuum or material plastic spin [7] $\boldsymbol{W}_{\mathrm{P}}$ via Eq. (4.15) and (5.2), *although a constitutive quantity*, is not constrained by the dissipation inequality.

To investigate the modelling of the $\vec{\boldsymbol{\alpha}}$ as structure tensors in a more concrete setting, assume for simplicity that $\vec{\boldsymbol{\alpha}}$ consists of the set $(\boldsymbol{Y}, \varepsilon)$, where $\boldsymbol{Y}$ is symmetric-tensor valued, and $\varepsilon$ scalar valued. In particular, the latter could represent, e.g., accumulated equivalent plastic strain, while the evolution of $\boldsymbol{Y}$ leads in general to both induced elastic and inelastic anisotropy. In this case, the free energy takes the form

$$\varphi(\boldsymbol{C}_{\mathrm{E}}, \vec{\boldsymbol{\alpha}}_{\mathrm{I}}, J_{\boldsymbol{P}}) = \varphi(\boldsymbol{C}_{\mathrm{E}}, \boldsymbol{Y}_{\mathrm{I}}, \varepsilon, J_{\boldsymbol{P}}), \tag{5.6}$$

with $\boldsymbol{Y}_{\mathrm{I}} := \boldsymbol{P}_{*}\boldsymbol{Y}$. In particular, if we model $\boldsymbol{Y}$ as $\boldsymbol{C}$-like, note that $\boldsymbol{P}_{*}\boldsymbol{Y} = \boldsymbol{P}^{-\mathrm{T}}\boldsymbol{Y}\boldsymbol{P}^{-1}$; on the other hand, $\boldsymbol{P}_{*}\boldsymbol{Y} = \boldsymbol{P}\boldsymbol{Y}\boldsymbol{P}^{\mathrm{T}}$ if $\boldsymbol{Y}$ is modelled as $\boldsymbol{C}^{-1}$-like. In any case, the isotropic form of $\varphi(\boldsymbol{C}_{\mathrm{E}}, \boldsymbol{Y}_{\mathrm{I}}, \varepsilon, J_{\boldsymbol{P}})$, given as usual by

$$\varphi(\boldsymbol{C}_{\mathrm{E}}, \boldsymbol{Y}_{\mathrm{I}}, \varepsilon, J_{\boldsymbol{P}}) = \varphi_{\mathrm{iso}}(I_{\boldsymbol{C}_{\mathrm{E}}}, I_{\boldsymbol{C}_{\mathrm{E}}^2}, I_{\boldsymbol{C}_{\mathrm{E}}^3}, I_{\boldsymbol{Y}_{\mathrm{I}}}, I_{\boldsymbol{Y}_{\mathrm{I}}^2}, I_{\boldsymbol{Y}_{\mathrm{I}}^3}, I_{\boldsymbol{C}_{\mathrm{E}}\boldsymbol{Y}_{\mathrm{I}}}, I_{\boldsymbol{C}_{\mathrm{E}}\boldsymbol{Y}_{\mathrm{I}}^2}, I_{\boldsymbol{C}_{\mathrm{E}}^2\boldsymbol{Y}_{\mathrm{I}}}, I_{\boldsymbol{C}_{\mathrm{E}}^2\boldsymbol{Y}_{\mathrm{I}}^2}, \varepsilon, J_{\boldsymbol{P}}). \tag{5.7}$$

This yields in turn that

$$2\varphi_{,\boldsymbol{C}_{\mathrm{E}}} = \mu_1 \boldsymbol{I} + \mu_2 \boldsymbol{C}_{\mathrm{E}} + \mu_3 \boldsymbol{C}_{\mathrm{E}}^2 + \mu_7 \boldsymbol{Y}_{\mathrm{I}} + \mu_8 \boldsymbol{Y}_{\mathrm{I}}^2 + \mu_9 \langle \boldsymbol{C}_{\mathrm{E}}, \boldsymbol{Y}_{\mathrm{I}} \rangle + \mu_{10} \langle \boldsymbol{C}_{\mathrm{E}}, \boldsymbol{Y}_{\mathrm{I}}^2 \rangle \tag{5.8}$$

for $2\varphi_{,\boldsymbol{C}_{\mathrm{E}}}$ which is clearly symmetric. Likewise, the derivative

$$2\varphi_{,\boldsymbol{Y}_{\mathrm{I}}} = \mu_4 \boldsymbol{I} + \mu_5 \boldsymbol{Y}_{\mathrm{I}} + \mu_6 \boldsymbol{Y}_{\mathrm{I}}^2 + \mu_7 \boldsymbol{C}_{\mathrm{E}} + \mu_8 \langle \boldsymbol{C}_{\mathrm{E}}, \boldsymbol{Y}_{\mathrm{I}} \rangle + \mu_9 \boldsymbol{C}_{\mathrm{E}}^2 + \mu_{10} \langle \boldsymbol{C}_{\mathrm{E}}^2, \boldsymbol{Y}_{\mathrm{I}} \rangle \tag{5.9}$$

of $\varphi$ with respect to $\boldsymbol{Y}_{\mathrm{I}}$ following from Eq. (5.7), is also symmetric. The material coefficients $\mu_{1-10}$ appearing in the last two expressions are given by

[6] This result is *formally* analogous to Noll's (1955) proof of the symmetry of the Cauchy stress on the basis of the material frame-indifference of the strain energy in the context of hyperelasticity; see, e.g., Truesdell and Noll (1965, Section 84).

[7] In his recent formulation of the concept of plastic spin, Dafalias (1998) refers to $\boldsymbol{W}_{\mathrm{P}}$ as the "plastic material" spin.

$$\begin{aligned}
&\mu_1 := 2\varphi_{\mathrm{iso,tr}(\boldsymbol{C}_\mathrm{E})}, \quad \mu_2 := 4\varphi_{\mathrm{iso,tr}(\boldsymbol{C}_\mathrm{E}^2)}, \quad \mu_3 := 6\varphi_{\mathrm{iso,tr}(\boldsymbol{C}_\mathrm{E}^3)},\\
&\mu_4 := 2\varphi_{\mathrm{iso,tr}(\boldsymbol{Y}_\mathrm{I})}, \quad \mu_5 := 4\varphi_{\mathrm{iso,tr}(\boldsymbol{Y}_\mathrm{I}^2)}, \quad \mu_6 := 6\varphi_{\mathrm{iso,tr}(\boldsymbol{Y}_\mathrm{I}^3)},\\
&\mu_7 := 2\varphi_{\mathrm{iso,tr}(\boldsymbol{C}_\mathrm{E}\boldsymbol{Y}_\mathrm{I})}, \quad \mu_8 := 2\varphi_{\mathrm{iso,tr}(\boldsymbol{C}_\mathrm{E}\boldsymbol{Y}_\mathrm{I}^2)}, \quad \mu_9 := 2\varphi_{\mathrm{iso,tr}(\boldsymbol{C}_\mathrm{E}^2\boldsymbol{Y}_\mathrm{I})},\\
&\mu_{10} := 2\varphi_{\mathrm{iso,tr}(\boldsymbol{C}_\mathrm{E}^2\boldsymbol{Y}_\mathrm{I}^2)}.
\end{aligned} \tag{5.10}$$

In particular, Eq. (5.8) implies that $\boldsymbol{M} = 2\boldsymbol{C}_\mathrm{E}\varphi_{,\boldsymbol{C}_\mathrm{E}}$ from Eq. (4.7) is not symmetric since $\boldsymbol{C}_\mathrm{E}$ and $\boldsymbol{Y}_\mathrm{I}$ do not commute, i.e., $[\boldsymbol{Y}_\mathrm{I}, \boldsymbol{C}_\mathrm{E}] \neq 0$. Likewise, Eq. (5.9) shows that $\boldsymbol{X}$ from Eq. (4.8), determined in part by either $-2\boldsymbol{Y}_\mathrm{I}\varphi_{,\boldsymbol{Y}_\mathrm{I}}$ (i.e., for $\boldsymbol{P}_*\boldsymbol{Y} = \boldsymbol{P}^{-\mathrm{T}}\boldsymbol{Y}\boldsymbol{P}^{-1}$) or $2\varphi_{,\boldsymbol{Y}_\mathrm{I}}\boldsymbol{Y}_\mathrm{I}$ (for $\boldsymbol{P}_*\boldsymbol{Y} = \boldsymbol{P}\boldsymbol{Y}\boldsymbol{P}^{\mathrm{T}}$), is as well not symmetric. On the other hand, the difference

$$\begin{aligned}
\boldsymbol{M} - \boldsymbol{X} &= 2\boldsymbol{C}_\mathrm{E}\varphi_{,\boldsymbol{C}_\mathrm{E}} + 2\boldsymbol{Y}_\mathrm{I}\varphi_{,\boldsymbol{Y}_\mathrm{I}} - J_{\boldsymbol{P}}\varphi_{,J_{\boldsymbol{P}}}\boldsymbol{I}\\
&= \mu_1\boldsymbol{C}_\mathrm{E} + \mu_2\boldsymbol{C}_\mathrm{E}^2 + \mu_3\boldsymbol{C}_\mathrm{E}^3 + \mu_4\boldsymbol{Y}_\mathrm{I} + \mu_5\boldsymbol{Y}_\mathrm{I}^2 + \mu_6\boldsymbol{Y}_\mathrm{I}^3 + \mu_7\langle\boldsymbol{C}_\mathrm{E}, \boldsymbol{Y}_\mathrm{I}\rangle + \mu_8\{\langle\boldsymbol{C}_\mathrm{E}, \boldsymbol{Y}_\mathrm{I}^2\rangle + \boldsymbol{Y}_\mathrm{I}\boldsymbol{C}_\mathrm{E}\boldsymbol{Y}_\mathrm{I}\}\\
&\quad + \mu_9\{\langle\boldsymbol{C}_\mathrm{E}^2, \boldsymbol{Y}_\mathrm{I}\rangle + \boldsymbol{C}_\mathrm{E}\boldsymbol{Y}_\mathrm{I}\boldsymbol{C}_\mathrm{E}\} + \mu_{10}\{\langle\boldsymbol{C}_\mathrm{E}^2, \boldsymbol{Y}_\mathrm{I}^2\rangle + \boldsymbol{C}_\mathrm{E}\boldsymbol{Y}_\mathrm{I}^2\boldsymbol{C}_\mathrm{E} + \boldsymbol{Y}_\mathrm{I}\boldsymbol{C}_\mathrm{E}^2\boldsymbol{Y}_\mathrm{I}\} - J_{\boldsymbol{P}}\varphi_{,J_{\boldsymbol{P}}}\boldsymbol{I},
\end{aligned} \tag{5.11}$$

holding for the case that $\boldsymbol{P}_*\boldsymbol{Y} = \boldsymbol{P}^{-\mathrm{T}}\boldsymbol{Y}\boldsymbol{P}^{-1}$, is clearly symmetric. Consequently, so is $\boldsymbol{\Sigma}_\mathrm{I} = \boldsymbol{M} - \boldsymbol{X}$ via Eq. (4.6), which then satisfies Eq. (5.2). Similarly,

$$\begin{aligned}
\boldsymbol{M} - \boldsymbol{X} &= 2\boldsymbol{C}_\mathrm{E}\varphi_{,\boldsymbol{C}_\mathrm{E}} - 2\varphi_{,\boldsymbol{Y}_\mathrm{I}}\boldsymbol{Y}_\mathrm{I} - J_{\boldsymbol{P}}\varphi_{,J_{\boldsymbol{P}}}\boldsymbol{I}\\
&= \mu_1\boldsymbol{C}_\mathrm{E} + \mu_2\boldsymbol{C}_\mathrm{E}^2 + \mu_3\boldsymbol{C}_\mathrm{E}^3 - \mu_4\boldsymbol{Y}_\mathrm{I} - \mu_5\boldsymbol{Y}_\mathrm{I}^2 - \mu_6\boldsymbol{Y}_\mathrm{I}^3 - \mu_8\boldsymbol{Y}_\mathrm{I}\boldsymbol{C}_\mathrm{E}\boldsymbol{Y}_\mathrm{I} + \mu_9\boldsymbol{C}_\mathrm{E}\boldsymbol{Y}_\mathrm{I}\boldsymbol{C}_\mathrm{E}\\
&\quad + \mu_{10}(\boldsymbol{C}_\mathrm{E}\boldsymbol{Y}_\mathrm{I}^2\boldsymbol{C}_\mathrm{E} - \boldsymbol{Y}_\mathrm{I}\boldsymbol{C}_\mathrm{E}^2\boldsymbol{Y}_\mathrm{I}) - J_{\boldsymbol{P}}\varphi_{,J_{\boldsymbol{P}}}\boldsymbol{I},
\end{aligned} \tag{5.12}$$

holding for the case $\boldsymbol{P}_*\boldsymbol{Y} = \boldsymbol{P}\boldsymbol{Y}\boldsymbol{P}^{\mathrm{T}}$, is symmetric as well. Note that this latter form for $\boldsymbol{\Sigma}_\mathrm{I} = \boldsymbol{M} - \boldsymbol{X}$ is simpler than Eq. (5.11) in the sense that the material coefficient $\mu_7$ plays no role in Eq. (5.12).

Consider next the constitutive form $\boldsymbol{\pi}_{\boldsymbol{Y}_\mathrm{I}} = \boldsymbol{\pi}_{\boldsymbol{Y}_\mathrm{I}}(\boldsymbol{Z}_\mathrm{I}, \varsigma, \boldsymbol{\Sigma}_\mathrm{I}, \boldsymbol{Y}_\mathrm{I}, \varepsilon, J_{\boldsymbol{P}})$ for $\boldsymbol{\pi}_{\boldsymbol{Y}_\mathrm{I}}$ corresponding to Eq. 5.6 with $\vec{\boldsymbol{\kappa}}_\mathrm{I} = (\boldsymbol{Z}_\mathrm{I}, \varsigma)$,

$$\boldsymbol{Z}_\mathrm{I} = -\varphi_{,\boldsymbol{Y}_\mathrm{I}} \tag{5.13}$$

and

$$\varsigma = -\varphi_{,\varepsilon} \tag{5.14}$$

from Eq. (4.4). In particular, we then have

$$\overset{\Delta}{\boldsymbol{Y}}_\mathrm{I} = h(\chi)\langle m\rangle\boldsymbol{\pi}_{\boldsymbol{Y}_\mathrm{I}} \tag{5.15}$$

for the evolution of $\boldsymbol{Y}_\mathrm{I}$ from Eq. (4.13). Here,

$$\overset{\Delta}{\boldsymbol{Y}}_\mathrm{I} = \boldsymbol{P}_*(\boldsymbol{P}^*\boldsymbol{Y}_\mathrm{I})^{\cdot} = \boldsymbol{P}(\boldsymbol{P}^{-1}\boldsymbol{Y}_\mathrm{I}\boldsymbol{P}^{-\mathrm{T}})^{\cdot}\boldsymbol{P}^{\mathrm{T}} = \dot{\boldsymbol{Y}}_\mathrm{I} - \boldsymbol{L}_\mathrm{P}\boldsymbol{Y}_\mathrm{I} - \boldsymbol{Y}_\mathrm{I}\boldsymbol{L}_\mathrm{P}^{\mathrm{T}} = \overset{\nabla}{\boldsymbol{Y}}_\mathrm{I} - \langle\boldsymbol{D}_\mathrm{P}, \boldsymbol{Y}_\mathrm{I}\rangle \tag{5.16}$$

holds for $\overset{\Delta}{\boldsymbol{Y}}_\mathrm{I}$ from Eq. (4.11) via Eq. (4.10), and first terms of Eqs. (5.4) and (5.5) when $\boldsymbol{Y}_\mathrm{I} = \boldsymbol{P}_*\boldsymbol{Y} = \boldsymbol{P}\boldsymbol{Y}\boldsymbol{P}^{\mathrm{T}}$,

$$\overset{\nabla}{\boldsymbol{Y}}_\mathrm{I} := \dot{\boldsymbol{Y}}_\mathrm{I} - [\boldsymbol{W}_\mathrm{P}, \boldsymbol{Y}_\mathrm{I}] \tag{5.17}$$

being the "plastic Jaumann" derivative. On the other hand, we would have

$$\overset{\Delta}{\boldsymbol{Y}}_\mathrm{I} = \boldsymbol{P}_*(\boldsymbol{P}^*\boldsymbol{Y}_\mathrm{I})^{\cdot} = \boldsymbol{P}^{-\mathrm{T}}(\boldsymbol{P}^{\mathrm{T}}\boldsymbol{Y}_\mathrm{I}\boldsymbol{P})^{\cdot}\boldsymbol{P}^{-1} = \dot{\boldsymbol{Y}}_\mathrm{I} + \boldsymbol{L}_\mathrm{P}^{\mathrm{T}}\boldsymbol{Y}_\mathrm{I} + \boldsymbol{Y}_\mathrm{I}\boldsymbol{L}_\mathrm{P} = \overset{\nabla}{\boldsymbol{Y}}_\mathrm{I} + \langle\boldsymbol{D}_\mathrm{P}, \boldsymbol{Y}_\mathrm{I}\rangle \tag{5.18}$$

via Eq. (5.17) when $\boldsymbol{Y}_\mathrm{I} = \boldsymbol{P}_*\boldsymbol{Y} = \boldsymbol{P}^{-\mathrm{T}}\boldsymbol{Y}\boldsymbol{P}^{-1}$. The set of evolution relations for this case is completed by the generalized flow rule Eq. (4.15) determining the evolution of $\boldsymbol{P}$, with $\boldsymbol{\pi}_{\boldsymbol{I}} = \boldsymbol{\pi}_{\boldsymbol{I}}(\boldsymbol{Z}_\mathrm{I}, \boldsymbol{\Sigma}_\mathrm{I}, \boldsymbol{Y}_\mathrm{I}, \varepsilon, J_{\boldsymbol{P}})$.

The effect of the evolution of $\boldsymbol{Y}$ in the form $\boldsymbol{Y}_{\mathrm{I}}$ as governed by Eq. (5.15) on the elastic and inelastic material behaviour in the context of Eq. (5.7) can be discussed in more detail with the help of the spectral representation

$$\boldsymbol{Y}_{\mathrm{I}} = y_1\boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + y_2\boldsymbol{n}_2 \otimes \boldsymbol{n}_2 + y_3\boldsymbol{n}_3 \otimes \boldsymbol{n}_3 = y_3\boldsymbol{I} + (y_1 - y_3)\boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + (y_2 - y_3)\boldsymbol{n}_2 \otimes \boldsymbol{n}_2 \tag{5.19}$$

of $\boldsymbol{Y}_{\mathrm{I}}$, the second form following from fact that $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ represents an orthonormal basis, such that $\boldsymbol{I} = \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + \boldsymbol{n}_2 \otimes \boldsymbol{n}_2 + \boldsymbol{n}_3 \otimes \boldsymbol{n}_3$ holds. On the basis of Eqs. (5.8) and (5.9) and this last form for $\boldsymbol{Y}_{\mathrm{I}}$, then, one sees that the eigenvectors of $\boldsymbol{Y}_{\mathrm{I}}$ can be interpreted as structure tensors for (in general) material orthotropy. On the other hand, unlike classical structure tensors, $\boldsymbol{Y}_{\mathrm{I}}$, and so its eigenvalues and eigenvectors, evolves, as such *inducing* this form of anisotropy in general. For example, consider an inelastic process starting at $t = 0$ with $\boldsymbol{P}(0) = \boldsymbol{I}$. In this case, $\boldsymbol{C}_{\mathrm{E}}(0) = \boldsymbol{C}(0)$ and $\boldsymbol{Y}_{\mathrm{I}}(0) = \boldsymbol{Y}(0)$ hold, the latter relation implying that $\boldsymbol{Y}(0)$ determines the initial anisotropy. In particular, if the initial elastic behaviour were effectively isotropic, $\boldsymbol{Y}(0) = \boldsymbol{I}$ would hold. This could be the case, for example, for a polycrystalline material in which the individual crystals (themselves being elastically anisotropic) are oriented randomly, i.e., before any significant texture [8] has developed. As shown by Eq. (5.19), such an effective initially isotropic behaviour would remain so if $\boldsymbol{Y}_{\mathrm{I}}(t > 0)$ were to evolve in such a fashion that all three of its eigenvalues $y_{1,2,3}$ remain equal during this evolution. Alternatively, if two of these were to remain equal, transverse isotropic material behaviour would ensue. In this way, the above formulation can (at least qualitatively) account for various special cases of the phenomenon of an anisotropic elastic material behaviour *induced* by inelastic processes in the material.

At this point, it is worth noting that the "standard" case of fixed (elastic or inelastic) anisotropic material behaviour is a special case of the current formulation; indeed, this case arises here when the $\vec{\boldsymbol{\alpha}}$ are assumed constant. For example, assuming the material behaviour is orthotropic with symmetry directions given by the constant referential orthonormal system $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$, we would have $\vec{\boldsymbol{\alpha}} = (\boldsymbol{n}_1 \otimes \boldsymbol{n}_1, \boldsymbol{n}_2 \otimes \boldsymbol{n}_2)$ (since $\boldsymbol{n}_3 \otimes \boldsymbol{n}_3 = \boldsymbol{I} - \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 - \boldsymbol{n}_2 \otimes \boldsymbol{n}_2$) in the simplest case. The free energy density then takes the form

$$\psi = (\boldsymbol{C}, \vec{\boldsymbol{\alpha}}, \boldsymbol{P}) = \varphi(\boldsymbol{P}_*\boldsymbol{C}, \boldsymbol{P}_*(\boldsymbol{n}_1 \otimes \boldsymbol{n}_1), \boldsymbol{P}_*(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2), J_P) \tag{5.20}$$

from Eq. (3.4a), with $\boldsymbol{P}_*\boldsymbol{n}_i = \boldsymbol{P}^{-\mathrm{T}}\boldsymbol{n}_i$ and so $\boldsymbol{P}_*(\boldsymbol{n}_i \otimes \boldsymbol{n}_i) = \boldsymbol{P}^{-\mathrm{T}}(\boldsymbol{n}_i \otimes \boldsymbol{n}_i)\boldsymbol{P}^{-1}$, or $\boldsymbol{P}_*\boldsymbol{n}_i = \boldsymbol{P}\boldsymbol{n}_i$ and so $\boldsymbol{P}_*(\boldsymbol{n}_i \otimes \boldsymbol{n}_i) = \boldsymbol{P}(\boldsymbol{n}_i \otimes \boldsymbol{n}_i)\boldsymbol{P}^{\mathrm{T}}$. As in the more general case of evolving $\vec{\boldsymbol{\alpha}}$, then, the stress measure $\boldsymbol{\Sigma}_{\mathrm{I}}$ as given by Eq. (4.6) fulfills Eq. (5.2), and is as such symmetric, again although neither $\boldsymbol{M}$ nor $\boldsymbol{X}$. Consequently, the problem identified and investigated by Lubliner (1986) as to the "correct" normal form of the flow rule in large deformation plasticity when the stress measure involved (i.e., here $\boldsymbol{\Sigma}_{\mathrm{I}}$) is not symmetric does not arise in the current formulation. This difficulty has forced many workers (e.g., Moran et al., 1990; Miehe, 1994; Maugin, 1994, 1995) in finite-deformation associated elastoplasticity to introduced more complicated forms for $\boldsymbol{D}_{\mathrm{P}}$ in the general anisotropic case (e.g., that $\boldsymbol{D}_{\mathrm{P}}$ is the symmetric part of $\boldsymbol{L}_{\mathrm{P}}$ with respect to the induced metric on the intermediate configuration). As shown by the current formulation, this is in fact unnecessary.

[8] In the case of texture development in polycrystalline materials, a "micromechanical" form for $\boldsymbol{Y}$ such as

$$\boldsymbol{Y} = \int_{S^2} g(\boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{n} \otimes \boldsymbol{n}\omega$$

might be relevant, analogous to the concept of alignment tensors for polymers or liquid crystals. Here, $g$ represents an orientation distribution function on the unit sphere $S^2$ with volume form $\omega$ involving a (at least integrable) distribution of glide plane normals $\boldsymbol{n}$ at $b \in B$. Although certainly a goal for future work, such a treatment, however, goes beyond the scope of the present work.

## 6. Plastic transformation and material uniformity

Up to this point, we have ignored the dependence of the constitutive relations, and in particular that of the free energy density $\psi$, on $b$, i.e., the possibility that distinct material elements of $B$ behave differently. In this section, we investigate the possibility that the material, i.e., $B$, is *uniform*, meaning that each $b \in B$ consists *locally* of the same material. As shown by Noll (1967) in the context of simple materials, it is then possible to model the dependence of the constitutive relations on $b \in B$ with the help of a *field* of local reference configurations on $B$, i.e., a different one for each $b \in B$. Equivalently, such a dependence can be modelled with the help of a field

$$\boldsymbol{H} : B \to \mathrm{Lin}^{+} \quad | \quad b \mapsto \boldsymbol{H} = \boldsymbol{H}(b) \tag{6.1}$$

of *transformations of local reference configuration* on $B$. To be more precise, $\boldsymbol{H}(b)$ transforms the non-uniform local reference configuration of each $b \in B$ into a uniform one. In the context of the modelling Eqs. (3.3a)–(3.3c) to Eqs. (3.4a)–(3.4c) of $\boldsymbol{P}$ as a transformation of local reference configuration, this leads in particular to the reduced form

$$\psi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, b) = \varphi_{\mathrm{U}}(\boldsymbol{H}(b)_{*}\boldsymbol{C}_{\mathrm{E}}, \boldsymbol{H}(b)_{*}\vec{\boldsymbol{\alpha}}_{\mathrm{I}}, J_{\boldsymbol{H}(b)\boldsymbol{P}}) = \varphi_{\mathrm{U}}((\boldsymbol{H}(b)\boldsymbol{P})_{*}\boldsymbol{C}, (\boldsymbol{H}(b)\boldsymbol{P})_{*}\vec{\boldsymbol{\alpha}}, J_{\boldsymbol{H}(b)\boldsymbol{P}}) \tag{6.2}$$

for $\psi(\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, b)$, where the subscript "U" stands for "uniform." Note that the material uniformity is represented here with respect to the transformed local reference configuration, i.e., with respect to the "intermediate" configuration. As with that of $\boldsymbol{P}$ in (3.4), the modelling of $\boldsymbol{H}$ as a material uniformity has been generalized slightly here (i.e., in comparison to (3.1)) in the sense that $\varphi_{\mathrm{U}}$ may depend non-linearly on $J_{\boldsymbol{H}}$.

Consider now the material "force density" $\nabla_{\boldsymbol{v}}\psi : B \to \mathbb{R}$ (having units of energy density or stress) in the direction $\boldsymbol{v} \in \mathscr{V}$, where $\nabla_{\boldsymbol{v}}$ represents the directional derivative operator in the direction $\boldsymbol{v} \in \mathscr{V}$ on the arbitrary global reference configuration of $B$ in question, i.e., that whose corresponding local configurations are transformed by $\boldsymbol{H}$ into uniform local configurations. In the context of Eq. (6.2), this force density takes the form

$$\nabla_{\boldsymbol{v}}\psi = \left\{ (\boldsymbol{H}_{*}\boldsymbol{C}_{\mathrm{E}})^{\mathrm{T}}_{,\boldsymbol{H}}[\varphi_{\mathrm{U},\boldsymbol{H}_{*}\boldsymbol{C}_{\mathrm{E}}}] + \sum_{i=1}^{n} (\boldsymbol{H}_{*}\boldsymbol{\alpha}_{\mathrm{I}i})^{\mathrm{T}}_{,\boldsymbol{H}}[\varphi_{\mathrm{U},\boldsymbol{H}_{*}\boldsymbol{\alpha}_{\mathrm{I}i}}] + \varphi_{\mathrm{U},J_{\boldsymbol{HP}}} J_{\boldsymbol{HP}} \boldsymbol{H}^{-\mathrm{T}} \right\} \nabla_{\boldsymbol{v}}\boldsymbol{H} \tag{6.3}$$

with respect to $\boldsymbol{H}$ via Eqs. (3.4a)–(3.4c). Now, with the help of the result

$$\psi_{,\boldsymbol{P}}\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{H}^{\mathrm{T}} \left\{ (\boldsymbol{H}_{*}\boldsymbol{C}_{\mathrm{E}})^{\mathrm{T}}_{,\boldsymbol{H}}[\varphi_{\mathrm{U},\boldsymbol{H}_{*}\boldsymbol{C}_{\mathrm{E}}}] + \sum_{i=1}^{n} (\boldsymbol{H}_{*}\boldsymbol{\alpha}_{\mathrm{I}i})^{\mathrm{T}}_{,\boldsymbol{H}}[\varphi_{\mathrm{U},\boldsymbol{H}_{*}\boldsymbol{\alpha}_{\mathrm{I}i}}] + \varphi_{\mathrm{U},J_{\boldsymbol{HP}}} J_{\boldsymbol{HP}} \boldsymbol{H}^{-\mathrm{T}} \right\} \tag{6.4}$$

from Eqs. (6.2) and (6.3) takes the alternative, compact form

$$\nabla_{\boldsymbol{v}}\psi = \psi_{,\boldsymbol{P}}\boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{H}^{-1}(\nabla_{\boldsymbol{v}}\boldsymbol{H}) = -\boldsymbol{\Sigma}_{\mathrm{I}} \cdot \boldsymbol{H}^{-1}(\nabla_{\boldsymbol{v}}\boldsymbol{H}) \tag{6.5}$$

in terms of $\boldsymbol{\Sigma}_{\mathrm{I}}$ via Eqs. (4.3). In the context of (6.2), then, this last result identifies $-\boldsymbol{\Sigma}_{\mathrm{I}}$ as one form of the so-called Eshelby stress tensor (see, e.g., Eshelby, 1951, 1970; Epstein and Maugin, 1990, 1996; Maugin, 1995). From Eq. (5.2), we see that this tensor is symmetric (i.e., with respect to the Euclidean metric), even in the case of anisotropic material behaviour, as discussed in the last sections. This is in contrast to the results of formulations which do not model the internal variables as structure tensors (e.g., Lubliner, 1986; Maugin, 1994), forcing one to consider the symmetry of $\boldsymbol{\Sigma}_{\mathrm{I}}$ with respect to other (i.e., non-Euclidean) metrics, and leading to a more complicated formulation. Further, on the basis Eqs. (4.6) and (6.5) show that the Eshelby stress is directly related to the Mandel and back stresses in the large-deformation inelastic context. As such, both elastic and inelastic processes contribute in general to $-\boldsymbol{\Sigma}_{\mathrm{I}}$, and so the material force density $\nabla_{\boldsymbol{v}}\psi$ field in a given direction $\boldsymbol{v} \in \mathscr{V}$.

## 7. Example: metal plasticity with hardening

To demonstrate the consequences of the current approach and formulation in more detail, it is useful to apply it to a particular case of practical interest. To this end, we consider in this section the the case of metal plasticity with non-linear isotropic, and Armstrong–Frederick kinematic, hardening. In particular, assume for simplicity that the material response is initially isotropic, that the influence of texture development, as well as damage, are negligible, and that kinematic hardening is of the Armstrong–Frederick type. On this basis, we have then in particular the form

$$\varphi(\boldsymbol{C}_{\mathrm{E}}, \boldsymbol{Y}_{\mathrm{I}}, \varepsilon) = \tfrac{1}{8}\lambda(\ln J_{\boldsymbol{C}_{\mathrm{E}}})^2 + \tfrac{1}{2}\mu\{(I_{\boldsymbol{C}_{\mathrm{E}}} - 3) - \ln J_{\boldsymbol{C}_{\mathrm{E}}}\} + \tfrac{1}{8}cI_{(\boldsymbol{Y}_{\mathrm{I}}-\boldsymbol{I})^2} + h(\varepsilon) \tag{7.1}$$

for the free energy density $\varphi$. Here, the first two terms, representing the "elastic part" of $\varphi$, model isotropic, compressible neo-Hooke type elastic behaviour. Further, $h(\varepsilon)$ represents the contribution of isotropic hardening to $\varphi$, $\varepsilon$ being the deformation-like internal variable conjugate to the yield stress. The remaining term in Eq. (7.1) represents the effect of kinematic hardening on $\varphi$, with $c$ the Armstrong–Frederick parameter mediating the (linear) growth of the back stress due to inelastic deformation. As indicated by the notation, the Lamé parameters $\lambda$, $\mu$, as well as $c$, are expressed with respect to the *original* (i.e., rather than transformed, or intermediate) local reference configuration, and are assumed constant.

The model (7.1) for the free energy density determines in particular to the form

$$\boldsymbol{M} = \tfrac{1}{2}\lambda(\ln J_{\boldsymbol{C}_{\mathrm{E}}})\boldsymbol{I} + \mu(\boldsymbol{C}_{\mathrm{E}} - \boldsymbol{I}) \tag{7.2}$$

for the Mandel stress from Eq. (4.7), and so that

$$\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{\mathrm{T}} = \tfrac{1}{2}\lambda(\ln J_{\boldsymbol{B}_{\mathrm{E}}})\boldsymbol{I} + \mu(\boldsymbol{B}_{\mathrm{E}} - \boldsymbol{I}) \tag{7.3}$$

for the Kirchhoff stress from Eq. (3.6), with $\boldsymbol{B}_{\mathrm{E}}$ the elastic left Cauchy–Green deformation measure. The small elastic strain versions of these last two relations are given by

$$\begin{aligned} \boldsymbol{M} &= \lambda I_{\ln \boldsymbol{U}_{\mathrm{E}}}\boldsymbol{I} + 2\mu \ln \boldsymbol{U}_{\mathrm{E}} + \mathrm{O}(2), \\ \boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{\mathrm{T}} &= \lambda I_{\ln \boldsymbol{V}_{\mathrm{E}}}\boldsymbol{I} + 2\mu \ln \boldsymbol{V}_{\mathrm{E}} + \mathrm{O}(2), \end{aligned} \tag{7.4}$$

respectively, where

$$\begin{aligned} \boldsymbol{C}_{\mathrm{E}} &= \exp(2\ln \boldsymbol{U}_{\mathrm{E}}) = \boldsymbol{I} + 2\ln \boldsymbol{U}_{\mathrm{E}} + \mathrm{O}(2), \\ \boldsymbol{B}_{\mathrm{E}} &= \exp(2\ln \boldsymbol{V}_{\mathrm{E}}) = \boldsymbol{I} + 2\ln \boldsymbol{V}_{\mathrm{E}} + \mathrm{O}(2), \end{aligned} \tag{7.5}$$

represent the corresponding geometric linearizations. Beyond these relations for the "elastic" stresses, Eq. (7.1) results in the forms

$$\boldsymbol{Z}_{\mathrm{I}} = -\varphi_{,\boldsymbol{Y}_{\mathrm{I}}} = \tfrac{1}{4}c(\boldsymbol{I} - \boldsymbol{Y}_{\mathrm{I}}), \tag{7.6a}$$

$$\varsigma = -\varphi_{,\varepsilon} = -h', \tag{7.6b}$$

for $\boldsymbol{Z}_{\mathrm{I}}$ and $\varsigma$ from Eqs. (5.13) and (5.14), respectively. Finally, from Eq. (7.1), we obtain

$$\boldsymbol{X} = \begin{cases} -2\boldsymbol{Y}_{\mathrm{I}}\varphi_{,\boldsymbol{Y}_{\mathrm{I}}} = \tfrac{1}{2}c\boldsymbol{Y}_{\mathrm{I}}(\boldsymbol{I} - \boldsymbol{Y}_{\mathrm{I}}) & \text{for } \boldsymbol{Y}_{\mathrm{I}} = \boldsymbol{P}^{-\mathrm{T}}\boldsymbol{Y}\boldsymbol{P}^{-1}, \\ 2\varphi_{,\boldsymbol{Y}_{\mathrm{I}}}\boldsymbol{Y}_{\mathrm{I}} = \tfrac{1}{2}c(\boldsymbol{Y}_{\mathrm{I}} - \boldsymbol{I})\boldsymbol{Y}_{\mathrm{I}} & \text{for } \boldsymbol{Y}_{\mathrm{I}} = \boldsymbol{P}\boldsymbol{Y}\boldsymbol{P}^{\mathrm{T}}, \end{cases} \tag{7.7}$$

for the back stress $\boldsymbol{X}$ via Eq. (4.8), the first holding for the case that $\boldsymbol{Y}$ is modelled as $\boldsymbol{C}$-like, and the second when $\boldsymbol{Y}$ is modelled as $\boldsymbol{C}^{-1}$-like.

The remainder of the formulation is based on an inelastic potential $p_{\mathrm{S}}$ of the form

$$p_{\mathrm{S}}(\boldsymbol{Z}_{\mathrm{I}}, \varsigma, \boldsymbol{\Sigma}_{\mathrm{I}}) = \phi_{\mathrm{S}}(\varsigma, \boldsymbol{\Sigma}_{\mathrm{I}}) + \frac{2b}{c} I_{Z_{\mathrm{I}}^2}, \tag{7.8}$$

expressed with respect to "stress-space". Here,

$$\phi_{\mathrm{S}}(\varsigma, \boldsymbol{\Sigma}_{\mathrm{I}}) = |\mathrm{dev}(\boldsymbol{\Sigma}_{\mathrm{I}})| - \sqrt{\frac{2}{3}}\sigma_{Y0} + \varsigma \leqslant 0 \tag{7.9}$$

represents the yield potential, $\sigma_{Y0}$ the initial yield stress, and $b$ the Armstrong-Frederick kinematic hardening saturation parameter. In addition, $h(\phi)\langle \ell \rangle = h(\chi)\langle m \rangle$ is identified with the standard plastic multiplier $\lambda$, i.e.,

$$\lambda = h(\phi)\langle \ell \rangle = h(\chi)\langle m \rangle. \tag{7.10}$$

From Eq. (7.8), one then obtains in particular the relation

$$\boldsymbol{D}_{\mathrm{P}} = \lambda p_{\mathrm{S},\boldsymbol{\Sigma}_{\mathrm{I}}} = \lambda \frac{\mathrm{dev}(\boldsymbol{\Sigma}_{\mathrm{I}})}{|\mathrm{dev}(\boldsymbol{\Sigma}_{\mathrm{I}})|} \tag{7.11}$$

for the flow rule via Eq. (4.15), the "associated" form $\mathrm{sym}(\boldsymbol{\pi}_{\boldsymbol{I}}) = p_{\mathrm{S},\boldsymbol{\Sigma}_{\mathrm{I}}} = \phi_{\mathrm{S},\boldsymbol{\Sigma}_{\mathrm{I}}}$ for $\mathrm{sym}(\boldsymbol{\pi}_{\boldsymbol{I}})$, and Eq. (5.4). Similarly,

$$\dot{\varepsilon} = \lambda p_{\mathrm{S},\varsigma} = \lambda \tag{7.12}$$

follows for the evolution of $\varepsilon$ assuming $\pi_{\varepsilon} = p_{\mathrm{S},\varsigma}$. Lastly, we have

$$\begin{aligned} \overset{\Delta}{\boldsymbol{Y}}_{\mathrm{I}} = \lambda p_{\mathrm{S},Z_{\mathrm{I}}} &= \lambda \frac{4b}{c} Z_{\mathrm{I}} \\ &= \lambda b(\boldsymbol{I} - \boldsymbol{Y}_{\mathrm{I}}) \\ &= |D_{\mathrm{P}}| b(\boldsymbol{I} - \boldsymbol{Y}_{\mathrm{I}}) \end{aligned} \tag{7.13}$$

for the evolution of $\boldsymbol{Y}_{\mathrm{I}}$ via the assumption $\boldsymbol{\pi}_{\boldsymbol{Y}_{\mathrm{I}}} = p_{\mathrm{S},Z_{\mathrm{I}}}$, Eqs. (7.6a) and (7.11). Substituting Eqs. (7.11)–(7.13) into the form of the dissipation rate density (5.3) applying to the current special case yields

$$\delta = \boldsymbol{Z}_{\mathrm{I}} \cdot \overset{\Delta}{\boldsymbol{Y}}_{\mathrm{I}} + \varsigma\dot{\varepsilon} + \boldsymbol{\Sigma}_{\mathrm{I}} \cdot D_{\mathrm{P}} = \lambda \left\{ \frac{2b}{c} I_{Z_{\mathrm{I}}^2} + \varsigma + |\mathrm{dev}(\boldsymbol{\Sigma}_{\mathrm{I}})| \right\} = \lambda \left\{ \frac{2b}{c} I_{Z_{\mathrm{I}}^2} + \sqrt{\frac{2}{3}}\sigma_{Y0} + \phi_{\mathrm{S}} \right\} \tag{7.14}$$

for the residual dissipation rate density via Eq. (7.8). Since $\delta$ is non-zero only when $\phi_{\mathrm{S}} = 0$ and $\lambda > 0$ from Eq. (7.10), $\delta \geqslant 0$ reduces to the requirement $2bI_{Z_{\mathrm{I}}^2}/c + \sigma_{Y0} \geqslant 0$ on the basis of Eq. (7.14). With $\sigma_{Y0} > 0$, this last requirement is satisfied sufficiently by $b/c \geqslant 0$.

To show that these results do in fact represent a large-strain generalization of the standard Armstrong–Frederick model which holds for small strain, consider the linearizations

$$\boldsymbol{Y} = \exp(\pm 2\boldsymbol{E}_{\boldsymbol{Y}}) = \boldsymbol{I} \pm 2\boldsymbol{E}_{\boldsymbol{Y}} + \mathrm{O}(2), \tag{7.15a}$$

$$\boldsymbol{Y}_{\mathrm{I}} = \exp(\pm 2\boldsymbol{E}_{\boldsymbol{Y}_{\mathrm{I}}}) = \boldsymbol{I} \pm 2\boldsymbol{E}_{\boldsymbol{Y}_{\mathrm{I}}} + \mathrm{O}(2), \tag{7.15b}$$

of $\boldsymbol{Y}_{\mathrm{I}}$ and $\boldsymbol{Y}$, with the plus (minus) sign holding when these are modelled as $\boldsymbol{C}$-like ($\boldsymbol{C}^{-1}$-like). In particular,

$$\boldsymbol{Y}_{\mathrm{I}} = \begin{cases} \boldsymbol{P}^{-\mathrm{T}}\boldsymbol{Y}\boldsymbol{P}^{-1} = \boldsymbol{I} + 2\boldsymbol{E}_{\boldsymbol{Y}} - 2\boldsymbol{E}_{\mathrm{P}} + \mathrm{O}(2) \\ \boldsymbol{P}\boldsymbol{Y}\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{I} - 2\boldsymbol{E}_{\boldsymbol{Y}} + 2\boldsymbol{E}_{\mathrm{P}} + \mathrm{O}(2) \end{cases} \tag{7.16}$$

then follow from Eq. (7.15a) and the result

$$\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{I} + 2\boldsymbol{E}_{\mathrm{P}} + \mathrm{O}(2). \tag{7.17}$$

Consequently, the relation

$$\boldsymbol{E}_{Y_{\mathrm{I}}} = \boldsymbol{E}_{\boldsymbol{Y}} - \boldsymbol{E}_{\mathrm{P}} \tag{7.18}$$

holds between $\boldsymbol{E}_{Y_{\mathrm{I}}}$ and $\boldsymbol{E}_{\boldsymbol{Y}}$ via Eq. (7.15b). From these, we have

$$\boldsymbol{Z}_{\mathrm{I}} = \mp\tfrac{1}{2}c\,\boldsymbol{E}_{Y_{\mathrm{I}}} + \mathrm{O}(2) = \pm\tfrac{1}{2}c\,(\boldsymbol{E}_{\mathrm{P}} - \boldsymbol{E}_{\boldsymbol{Y}}) + \mathrm{O}(2) \tag{7.19}$$

for $\boldsymbol{Z}_{\mathrm{I}}$ in the respective cases from Eq. (7.6a) via Eq. (7.18). On the other hand, the two forms (7.7) for $\boldsymbol{X}$ are equal to $\mathrm{O}(2)$, i.e.,

$$\boldsymbol{X} = -c\,\boldsymbol{E}_{Y_{\mathrm{I}}} + \mathrm{O}(2) = c\,(\boldsymbol{E}_{\mathrm{P}} - \boldsymbol{E}_{\boldsymbol{Y}}) + \mathrm{O}(2) \tag{7.20}$$

holds in this case. Finally, we have

$$\overset{\Delta}{\boldsymbol{Y}}_{\mathrm{I}} = \dot{\boldsymbol{Y}}_{\mathrm{I}} - [\boldsymbol{W}_{\mathrm{P}}, \boldsymbol{Y}_{\mathrm{I}}] \pm \langle \boldsymbol{D}_{\mathrm{P}}, \boldsymbol{Y}_{\mathrm{I}} \rangle = \pm 2\,\dot{\boldsymbol{E}}_{Y_{\mathrm{I}}} \pm 2\,\boldsymbol{D}_{\mathrm{P}} + \mathrm{O}(2) \tag{7.21}$$

and

$$\overset{\Delta}{\boldsymbol{Y}}_{\mathrm{I}} = \boldsymbol{P}_{*}\,\dot{\boldsymbol{Y}} = \pm 2\,\dot{\boldsymbol{E}}_{\boldsymbol{Y}} + \mathrm{O}(2) \tag{7.22}$$

from Eqs. (5.16)–(5.18), (7.15a) and (7.15b). Combining these with Eq. (7.18) yields the identification

$$\boldsymbol{D}_{\mathrm{P}} = \dot{\boldsymbol{E}}_{\mathrm{P}} + \mathrm{O}(2), \tag{7.23}$$

a result obtainable directly from Eq. (7.17) as well. So, on the basis of Eqs. (7.21) and (7.23), Eq. (7.12) reduces to

$$-\dot{\boldsymbol{E}}_{Y_{\mathrm{I}}} = \dot{\boldsymbol{E}}_{\mathrm{P}} + |\dot{\boldsymbol{E}}_{\mathrm{P}}|\,b\,\boldsymbol{E}_{Y_{\mathrm{I}}} \ \text{ or } \ \dot{\boldsymbol{E}}_{\boldsymbol{Y}} = |\dot{\boldsymbol{E}}_{\mathrm{P}}|\,b\,(\boldsymbol{E}_{\mathrm{P}} - \boldsymbol{E}_{\boldsymbol{Y}}) \tag{7.24}$$

via Eqs. (7.15b) and (7.18). Combined then with the first term of Eq. (7.20), the first of these last relation yields in fact the standard, i.e., small strain, form of the Armstrong–Frederick evolution relation for $\boldsymbol{X}$. Note that $\boldsymbol{W}_{\mathrm{P}}$ is assumed $\mathrm{O}(2)$ in the above.

## 8. Numerical aspects: consistent tangent

The implementation of the current approach to inelastic material behaviour in a numerical simulation of such behaviour in engineering structures via, e.g., the finite element method, is based among other things on the formulation of the so-called consistent tangent. The purpose of this section is to derive the general form of this tangent for the current approach in the context of local standard backward-Euler integration of the material model.

To begin, let $B_{\mathrm{r}} \subset E$ be a reference configuration of the material body $B$ in three-dimensional Euclidean point space $E$. The deformation of $B_{\mathrm{r}}$ into any other configuration $B_{\mathrm{i}} \subset E$ of this body in $E$ can be represented by a local diffeomorphism

$$\boldsymbol{\xi}_i : B_{\mathrm{r}} \to B_i \quad | \quad \boldsymbol{x}_{\mathrm{r}} \mapsto \boldsymbol{x}_i = \boldsymbol{\xi}_i(\boldsymbol{x}_{\mathrm{r}}). \tag{8.1}$$

This is related to the more familiar displacement field $\boldsymbol{u}_i : B_{\mathrm{r}} \to \mathscr{V}$ via

$$\boldsymbol{\xi}_i = 1_{B_{\mathrm{r}}} + \boldsymbol{u}_i, \tag{8.2}$$

with $1_{B_{\mathrm{r}}} : B_{\mathrm{r}} \to B_{\mathrm{r}} \mid \boldsymbol{x}_{\mathrm{r}} \mapsto \boldsymbol{x}_{\mathrm{r}} = 1_{B_{\mathrm{r}}}(\boldsymbol{x}_{\mathrm{r}})$ the identity mapping. In addition, we have the deformation gradient

$$\boldsymbol{F}_i := \nabla_{\mathrm{r}}\boldsymbol{\xi}_i = \nabla_{\mathrm{r}} 1_{B_{\mathrm{r}}} + \nabla_{\mathrm{r}}\boldsymbol{u}_i = \boldsymbol{I} + \nabla_{\mathrm{r}}\boldsymbol{u}_i \tag{8.3}$$

corresponding to $\boldsymbol{\xi}_i$. In this section, $\nabla_i$ represents the gradient operator with respect to $B_i$.

Consider next equilibrium momentum balance. As usual, any deformation $\boldsymbol{\xi}_{\mathrm{s}} : B_{\mathrm{r}} \to B_{\mathrm{s}}$ satisfies this balance in the weak sense when the functional [9]

$$\Gamma(\boldsymbol{\xi}_{\mathrm{s}}, \boldsymbol{\zeta}) := \int_{B_{\mathrm{r}}} (\nabla_{\mathrm{r}} \boldsymbol{\xi}_{\mathrm{s}}) \boldsymbol{S}(\nabla_{\mathrm{r}} \boldsymbol{\xi}_{\mathrm{s}}) \cdot \nabla_{\mathrm{r}} \boldsymbol{\zeta} - \int_{B_{\mathrm{r}}} \boldsymbol{b}(\boldsymbol{\xi}_{\mathrm{s}}, \nabla_{\mathrm{r}} \boldsymbol{\xi}_{\mathrm{s}}) \cdot \boldsymbol{\zeta} - \int_{\partial B_{\mathrm{r}}} \boldsymbol{t}(\boldsymbol{\xi}_{\mathrm{s}}, \nabla_{\mathrm{r}} \boldsymbol{\xi}_{\mathrm{s}}) \cdot \boldsymbol{\zeta} \tag{8.4}$$

vanishes for all admissible virtual deformations $\boldsymbol{\zeta} : B_{\mathrm{r}} \to \mathscr{V}$, i.e., those vanishing on the parts of the boundary $\partial B_{\mathrm{r}}$ of $B_{\mathrm{r}}$ where the displacements are prescribed. Here, $\boldsymbol{b}$ represents the body force density, and $\boldsymbol{t}$ the traction on $\partial B_{\mathrm{r}}$. As usual, the "consistent" linearization of Eq. (8.4) in the context of the Newton–Ralphson method (e.g., Hughes and Pister, 1979) is based in part on the fact that any two deformations $\boldsymbol{\xi}_i : B_{\mathrm{r}} \to B_i$ and $\boldsymbol{\xi}_j : B_{\mathrm{r}} \to B_j$ are related to each other via

$$\boldsymbol{\xi}_i = \boldsymbol{\xi}_j + \boldsymbol{u}_{ij}, \tag{8.5}$$

$\boldsymbol{u}_{ij}$ being the (relative) displacement from $B_i$ to $B_j$ expressed as a field on $B_{\mathrm{r}}$. These induce in turn via differentiation the connection

$$\boldsymbol{F}_i = \boldsymbol{F}_j + \nabla_{\mathrm{r}} \boldsymbol{u}_{ij} \tag{8.6}$$

between the corresponding deformation gradients. With the help of these, we obtain the linearization

$$\Gamma(\boldsymbol{\xi}_i, \boldsymbol{\zeta}) = \Gamma(\boldsymbol{\xi}_j, \boldsymbol{\zeta}) + \mathscr{D}_{\xi} \Gamma(\boldsymbol{\xi}_j, \boldsymbol{\zeta}) \cdot \boldsymbol{u}_{ij} + \mathrm{O}(2) \tag{8.7}$$

of Eq. (8.4), $\mathscr{D}_{\xi} \Gamma(\boldsymbol{\xi}_j, \boldsymbol{\zeta}) \cdot \boldsymbol{u}_{ij}$ being the partial functional directional derivative of $\Gamma(\cdot, \boldsymbol{\zeta})$ in the direction $\boldsymbol{u}_{ij}$. Assuming for simplicity that the traction $\boldsymbol{t}$ and body force $\boldsymbol{b}$ are in fact independent of deformation (i.e., dead loads), we have

$$\mathscr{D}_{\xi} \Gamma(\boldsymbol{\xi}_j, \boldsymbol{\zeta}) \cdot \boldsymbol{u}_{ij} = \int_{B_{\mathrm{r}}} D(\boldsymbol{FS})(\boldsymbol{F}_j)[\nabla_{\mathrm{r}} \boldsymbol{u}_{ij}] \cdot \nabla_{\mathrm{r}} \boldsymbol{\zeta}. \tag{8.8}$$

Here,

$$D(\boldsymbol{FS})(\boldsymbol{F})[\boldsymbol{A}] = \boldsymbol{A}\boldsymbol{S}(\boldsymbol{F}) + \boldsymbol{F}\,(D\boldsymbol{S})(\boldsymbol{F})[\boldsymbol{A}] \tag{8.9}$$

represents the algorithmic linearization of the first Piola–Kirchhoff stress with respect to $\boldsymbol{F}$ in the direction $\boldsymbol{A}$, with

$$D\boldsymbol{S}(\boldsymbol{F})\,[\boldsymbol{A}] = D\boldsymbol{S}(\boldsymbol{C}(\boldsymbol{F}))\,D\boldsymbol{C}(\boldsymbol{F})\,[\boldsymbol{A}] = D\boldsymbol{S}(\boldsymbol{C}(\boldsymbol{F}))\,[\boldsymbol{F}^{\mathrm{T}}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{F}] \tag{8.10}$$

that of the second Piola–Kirchhoff stress with respect to $\boldsymbol{F}$. Using these results in Eq. (8.8) yields

$$\mathscr{D}_{\xi} \Gamma(\boldsymbol{\xi}_j, \boldsymbol{\zeta}) \cdot \boldsymbol{u}_{ij} = \int_{B_{\mathrm{r}}} (\nabla_j \boldsymbol{u}_{ij})\, \boldsymbol{F}_j \boldsymbol{S}(\boldsymbol{C}_j)\, \boldsymbol{F}_j^{\mathrm{T}} \cdot \nabla_j \boldsymbol{\zeta} + \int_{B_{\mathrm{r}}} 2\, \boldsymbol{F}_j D\boldsymbol{S}(\boldsymbol{C}_j)\, [\boldsymbol{F}_j^{\mathrm{T}} \mathrm{sym}(\nabla_j \boldsymbol{u}_{ij}) \boldsymbol{F}_j] \boldsymbol{F}_j^{\mathrm{T}} \cdot \nabla_j \boldsymbol{\zeta}, \tag{8.11}$$

in terms of the Kirchhoff stress $\boldsymbol{FSF}^{\mathrm{T}}$ and the consistent, algorithmic linearization $D\boldsymbol{S}(\boldsymbol{C})$ of $\boldsymbol{S}(\boldsymbol{C})$ with respect to $\boldsymbol{C}$.

In the internal variable context, $D\boldsymbol{S}(\boldsymbol{C})$ depends as usual on the inelastic material model and the choice of numerical integration. For example, in the current context, standard backward-Euler integration of the evolution relation (2.5) for each $\boldsymbol{\epsilon}_i$, $i = 1, \ldots, n$, together with the yield condition $\phi \leqslant 0$, yields the algorithmic system

$$\begin{bmatrix} r_{\vec{\epsilon}} \\ r_{\gamma} \end{bmatrix} (\boldsymbol{C}, \vec{\boldsymbol{\epsilon}}, \gamma) = \begin{bmatrix} \boldsymbol{0} \\ 0 \end{bmatrix} \tag{8.12}$$

[9] The volume d$v$ and surface d$a$ elements are dispensed with in the corresponding integrals to follow for notational simplicity.

to solve, with $r_{\vec{\epsilon}} := (r_{\epsilon_1}, \ldots, r_{\epsilon_n})$,

$$r_{\epsilon_i}(\boldsymbol{C}, \vec{\epsilon}, \gamma) := \epsilon_i - \epsilon_{i0} - \gamma \varpi_{\epsilon_i}(\vec{\sigma}(\boldsymbol{C}, \vec{\epsilon}), \vec{\epsilon}), \quad i = 1, \ldots, n, \tag{8.13a}$$

$$r_{\gamma}(\boldsymbol{C}, \vec{\epsilon}, \gamma) := \phi_{\mathrm{S}}(\vec{\sigma}(\boldsymbol{C}, \vec{\epsilon}), \vec{\epsilon}), \tag{8.13b}$$

and $\gamma := \lambda \Delta t = h(\phi) \langle \ell \rangle \Delta t$. The systems (8.12), (8.13a) and (8.13b) is solved locally (i.e., at fixed $\boldsymbol{C}$) as usual via linearization and Newton iteration. This involves solution of the system

$$\begin{array}{lcll} r_{\vec{\epsilon},\vec{\epsilon}}[\Delta\vec{\epsilon}] & + & r_{\vec{\epsilon},\gamma}\Delta\gamma & = & -r_{\vec{\epsilon}}, \\ r_{\gamma,\vec{\epsilon}} \cdot \Delta\vec{\epsilon} & + & r_{\gamma,\gamma}\Delta\gamma & = & -r_{\gamma}, \end{array} \tag{8.14}$$

at each step of the local iteration in the context of, e.g., the elastic predictor-plastic corrector approach (e.g., Simo and Hughes, 1987). Here,

$$(r_{\vec{\epsilon},\vec{\epsilon}}) = \begin{bmatrix} r_{\epsilon_1,\epsilon_1} & \cdots & r_{\epsilon_1,\epsilon_n} \\ \vdots & \ddots & \vdots \\ r_{\epsilon_n,\epsilon_1} & \cdots & r_{\epsilon_n,\epsilon_n} \end{bmatrix} = \begin{bmatrix} l_1 - \gamma \varpi_{\epsilon_1,\epsilon_1} & \cdots & -\gamma \varpi_{\epsilon_1,\epsilon_n} \\ \vdots & \ddots & \vdots \\ -\gamma \varpi_{\epsilon_n,\epsilon_1} & \cdots & l_n - \gamma \varpi_{\epsilon_n,\epsilon_n} \end{bmatrix} \tag{8.15}$$

follows from Eq. (8.13a), with $l_i$ the identity mapping on the inner product space on which $\epsilon_i$ is defined. As shown by, e.g., Simo and Hughes (1987) (see also, e.g., Hartmann, 1993; Hartmann and Haupt, 1993), a system such as Eq. (8.14) can in fact be solved to obtain

$$\Delta\gamma = \frac{r_{\gamma,\vec{\epsilon}} \cdot (r_{\vec{\epsilon},\vec{\epsilon}})^{-1} r_{\vec{\epsilon}} - r_{\gamma}}{r_{\gamma,\gamma} - r_{\gamma,\vec{\epsilon}} \cdot (r_{\vec{\epsilon},\vec{\epsilon}})^{-1} r_{\vec{\epsilon},\gamma}} = \frac{\sum_{i,j=1}^{n} r_{\gamma,\epsilon_i} \cdot (r_{\vec{\epsilon},\vec{\epsilon}})^{-1}_{ij} r_{\epsilon_j} - r_{\gamma}}{\sum_{i,j=1}^{n} r_{\gamma,\epsilon_i} \cdot (r_{\vec{\epsilon},\vec{\epsilon}})^{-1}_{ij} \varpi_{\epsilon_j}} \tag{8.16}$$

and

$$\Delta\vec{\epsilon} = -(r_{\vec{\epsilon},\vec{\epsilon}})^{-1}[r_{\vec{\epsilon},\gamma}\Delta\gamma + r_{\vec{\epsilon}}] = (r_{\vec{\epsilon},\vec{\epsilon}})^{-1}[\varpi_{\vec{\epsilon}}\Delta\gamma - r_{\vec{\epsilon}}] \tag{8.17}$$

for the updates of the local Newton iteration with $r_{\epsilon_i,\gamma} = -\varpi_{\epsilon_i}$ and $r_{\gamma,\gamma} = 0$ via Eqs. (8.13a) and (8.13b). In particular, note that $r_{\gamma,\epsilon_i}$ takes the form

$$r_{\gamma,\epsilon_i} = \phi_{,\epsilon_i} - \sum_{j=1}^{n} \psi_{,\epsilon_i\epsilon_j}[\phi_{,\sigma_j}] \tag{8.18}$$

via Eqs. (2.12), (8.13a) and (8.13b). The existence of Eqs. (8.16) and (8.17) as solutions of Eq. (8.14) depends on the determinant

$$\det \begin{bmatrix} r_{\vec{\epsilon},\vec{\epsilon}} & r_{\vec{\epsilon},\gamma} \\ r_{\gamma,\vec{\epsilon}} & r_{\gamma,\gamma} \end{bmatrix} = \det(r_{\vec{\epsilon},\vec{\epsilon}})\, r_{\gamma,\vec{\epsilon}} \cdot (r_{\vec{\epsilon},\vec{\epsilon}})^{-1} \varpi_{\vec{\epsilon}} \tag{8.19}$$

of the partitioned coefficient matrix in Eq. (8.14) being non-zero (e.g., Press et al., 1992, Chapter 2).

With the local solution in hand, that of the momentum balance Eq. (8.4) via linearization equation (8.7) and Newton–Ralphson iteration for variable $\boldsymbol{C}$ is then contingent on establishing the algorithmic dependence of the $\vec{\epsilon}$ on $\boldsymbol{C}$. This can be done implicitly via solution of the system

$$\begin{array}{lcll} r_{\vec{\epsilon},\vec{\epsilon}}[\mathrm{d}\vec{\epsilon}] & + & r_{\vec{\epsilon},\gamma}\,\mathrm{d}\gamma & = & -r_{\vec{\epsilon},\boldsymbol{C}}[\mathrm{d}\boldsymbol{C}], \\ r_{\gamma,\vec{\epsilon}} \cdot \mathrm{d}\vec{\epsilon} & + & r_{\gamma,\gamma}\,\mathrm{d}\gamma & = & -r_{\gamma,\boldsymbol{C}} \cdot \mathrm{d}\boldsymbol{C}, \end{array} \tag{8.20}$$

following from Eqs. (8.12), Eqs. (8.13a) and (8.13b) for variable $\boldsymbol{C}$. With $\mathrm{d}\gamma = D\gamma(\boldsymbol{C})\mathrm{d}\boldsymbol{C}$ and $\mathrm{d}\vec{\epsilon} = D\vec{\epsilon}(\boldsymbol{C})[\mathrm{d}\boldsymbol{C}]$, the system (8.20) can, like Eq. (8.14), be solved to obtain

$$D\gamma = \frac{-1}{r_{\gamma,\vec{\epsilon}} \cdot (r_{\vec{\epsilon},\vec{\epsilon}})^{-1}\boldsymbol{\varpi}_{\vec{\epsilon}}} \left\{ \gamma(\boldsymbol{\varpi}_{\vec{\epsilon},\boldsymbol{C}})^{\mathrm{T}}(r_{\vec{\epsilon},\vec{\epsilon}})^{-\mathrm{T}}[r_{\gamma,\vec{\epsilon}}] + r_{\gamma,\boldsymbol{C}} \right\} \tag{8.21}$$

and

$$D\vec{\boldsymbol{\epsilon}} = (r_{\vec{\epsilon},\vec{\epsilon}})^{-1}[\boldsymbol{\varpi}_{\vec{\epsilon}} \otimes D\gamma + \gamma\boldsymbol{\varpi}_{\vec{\epsilon},\boldsymbol{C}}]. \tag{8.22}$$

Note that

$$r_{\gamma,\boldsymbol{C}} = \phi_{,\boldsymbol{C}} = -\sum_{i=1}^{n} \psi_{,\boldsymbol{C}\boldsymbol{\epsilon}_i}[\phi_{,\boldsymbol{\sigma}_i}] \tag{8.23}$$

as well as

$$\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i,\boldsymbol{C}} = -\sum_{j=1}^{n} \boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i,\boldsymbol{\sigma}_j}\psi_{,\boldsymbol{\epsilon}_j\boldsymbol{C}} \tag{8.24}$$

hold from Eqs. (8.13a) and (8.13b). The result (8.22) for $D\vec{\boldsymbol{\epsilon}} = (D\boldsymbol{\epsilon}_1, \ldots, D\boldsymbol{\epsilon}_n)$ in turn determines then the "consistent" algorithmic form

$$D\boldsymbol{S} = \boldsymbol{S}_{,\boldsymbol{C}} + \sum_{i=1}^{n} \boldsymbol{S}_{,\boldsymbol{\epsilon}_i} D\boldsymbol{\epsilon}_i = 2\psi_{,\boldsymbol{C}\boldsymbol{C}} + \sum_{i=1}^{n} 2\psi_{,\boldsymbol{C}\boldsymbol{\epsilon}_i} D\boldsymbol{\epsilon}_i \tag{8.25}$$

for $D\boldsymbol{S}(\boldsymbol{C})$ in the context of local backward-Euler integration. This can be compared with the so-called elastoplastic tangent

$$\boldsymbol{C}_{\mathrm{EP}} = 2\psi_{,\boldsymbol{C}\boldsymbol{C}} + \sum_{i=1}^{n} 2\psi_{,\boldsymbol{C}\boldsymbol{\epsilon}_i}(\boldsymbol{\varpi}_{\boldsymbol{\epsilon}_i} \otimes \boldsymbol{N}) \tag{8.26}$$

holding for $\phi = 0$ and $\ell > 0$, obtained via time-differentiation of Eq. (2.9), and use of Eq. (2.5). As discussed in Section 2, $\boldsymbol{N}$ is given by either Eq. (2.13) when Eq. (2.14) is assumed to hold, or by Eq. (2.15) when we identify $h(\phi)\langle\ell\rangle$ with the standard plastic multiplier $\lambda$ as done in Eq. (7.9) in the last section. In any case, the consistent form Eq. (8.25) of $D\boldsymbol{S}(\boldsymbol{C})$ yields that of $\mathscr{D}_{\xi}\Gamma(\boldsymbol{\xi},\boldsymbol{\zeta})$ via (8.8), and so the numerical consistent tangent via FE-discretization of Eqs. (8.4),(8.7) and (8.8) as usual.

Consider as an example of the system Eqs. (8.13a) and (8.13b) that pertaining to the model for metal plasticity discussed in the last section. The basic potentials are given by

$$\begin{aligned} &\psi(\boldsymbol{C},\boldsymbol{Y},\varepsilon,\boldsymbol{P}) = \varphi(\boldsymbol{P}_*\boldsymbol{C},\boldsymbol{P}_*\boldsymbol{Y},\varepsilon),\\ &P(\boldsymbol{C},\boldsymbol{Y},\varepsilon,\boldsymbol{P}) = P_s(\boldsymbol{Z}_{\mathrm{I}}(\boldsymbol{P}_*\boldsymbol{Y}),\varsigma(\varepsilon),\Sigma_{\mathrm{I}}(\boldsymbol{P}_*\boldsymbol{C},\boldsymbol{P}_*\boldsymbol{Y})), \end{aligned} \tag{8.27}$$

from Eqs. (7.1), (7.2), (7.6a), (7.6b), (7.7) and (7.8). Further, Eq. (8.12) holds with $\vec{r} := (r_{\boldsymbol{Y}}, r_{\varepsilon}, r_{\boldsymbol{P}}, r_{\gamma})$ and

$$r_{\boldsymbol{Y}}(\boldsymbol{C},\boldsymbol{Y},\varepsilon,\boldsymbol{P},\gamma) = \boldsymbol{Y} - \boldsymbol{Y}_0 - \gamma b(\boldsymbol{P}^{-1}\boldsymbol{P}^{-\mathrm{T}} - \boldsymbol{Y}), \tag{8.28a}$$

$$r_{\varepsilon}(\boldsymbol{C},\boldsymbol{Y},\varepsilon,\boldsymbol{P},\gamma) = \varepsilon - \varepsilon_0 - \gamma, \tag{8.28b}$$

$$r_{\boldsymbol{P}}(\boldsymbol{C},\boldsymbol{Y},\varepsilon,\boldsymbol{P},\gamma) = \boldsymbol{P}r_{\boldsymbol{P}}(\boldsymbol{C},\boldsymbol{Y},\varepsilon,\boldsymbol{P},\gamma) = \boldsymbol{P} - \exp\{\gamma\mathrm{dev}(\boldsymbol{\Sigma}_{\mathrm{I}})/|\mathrm{dev}(\boldsymbol{\Sigma}_{\mathrm{I}})| + \Delta t\boldsymbol{W}_{\mathrm{P}}\}\boldsymbol{P}_0, \tag{8.28c}$$

$$r_{\gamma}(\boldsymbol{C},\boldsymbol{Y},\varepsilon,\boldsymbol{P},\gamma) = r_{\gamma}(\boldsymbol{C},\boldsymbol{Y},\varepsilon,\boldsymbol{P},\gamma) = \phi_{\mathrm{S}}(\varsigma(\varepsilon),\Sigma_{\mathrm{I}}(\boldsymbol{P}_*\boldsymbol{C},\boldsymbol{P}_*\boldsymbol{Y})). \tag{8.28d}$$

In particular, the form of Eq. (8.28a) presumes that $\boldsymbol{Y}$ is modelled as $\boldsymbol{C}^{-1}$-like, i.e., $\boldsymbol{Y}_{\mathrm{I}} = \boldsymbol{P}_*\boldsymbol{Y} = \boldsymbol{P}\boldsymbol{Y}\boldsymbol{P}^{\mathrm{T}}$ holds. Here, we have,

$$\boldsymbol{\Sigma}_{\mathrm{I}}(\boldsymbol{P}^{-\mathrm{T}}\boldsymbol{C}\boldsymbol{P}^{-1},\boldsymbol{P}\boldsymbol{Y}\boldsymbol{P}^{\mathrm{T}}) = \tfrac{1}{2}\lambda(\ln J_{\boldsymbol{P}^{-\mathrm{T}}\boldsymbol{C}\boldsymbol{P}^{-1}})\boldsymbol{I} + \mu(\boldsymbol{P}^{-\mathrm{T}}\boldsymbol{C}\boldsymbol{P}^{-1} - \boldsymbol{I}) - \tfrac{1}{2}c(\boldsymbol{P}\boldsymbol{Y}\boldsymbol{P}^{\mathrm{T}} - \boldsymbol{I})\boldsymbol{P}\boldsymbol{Y}\boldsymbol{P}^{\mathrm{T}}, \tag{8.29}$$

from Eqs. (4.6), (7.2) and (7.7), $\boldsymbol{C}_{\mathrm{P}} := \boldsymbol{P}^{\mathrm{T}}\boldsymbol{P}$ being the plastic right Cauchy–Green deformation. From a somewhat different point of view than that taken here, possibilities for the material plastic spin $\boldsymbol{W}_{\mathrm{P}}$ have been investigated in Svendsen et al. (1998) in the context of small elastic strain and simple shear. In addition, a model similar to that above had been employed by Arndt et al. (1997) in a finite-element-based numerical simulation of non-linear combined hardening and ductile damage in fracture mechanics specimens.

## Acknowledgements

I thank Albrecht Bertram, Thomas Böhlke, Peter Haupt, Gérard Maugin, Stefanie Reese, Rainer Sievert, and Charalampos Tsakmakis for useful discussions on many aspects of this work.

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